The Understanding of Additive and Multiplicative Arithmetic Concepts

Katherine M. Robinson
Department of Psychology, Campus College, University of Regina, Regina, SK, Canada

INTRODUCTION

In a report requested by the U.S. National Research Council on the state of research and knowledge on children's mathematics learning, Kilpatrick, Swafford, and Findell (2001) concluded that: “The more mathematical concepts they understand, the more sensible mathematics becomes” (p. 131). According to Kilpatrick et al., conceptual understanding helps organize mathematical information into a coherent whole, helps relate new knowledge to old, enhances retention of mathematical knowledge and information, and therefore provides students with the ability to connect concepts and procedures. Conceptual knowledge can also facilitate learning because it enables children to identify similarities across problem solving situations even when the situations appear to be, at least on the surface, unrelated. Children with good conceptual knowledge have deeply organized their knowledge into a strong network of facts and principles (Kilpatrick et al., 2001).

One of the first studies in psychology to examine children's understanding of an arithmetic concept was reported by Starkey and Gelman (1982). Starkey and Gelman based their study on Piaget's (1952) tenet that children can only truly understand addition and subtraction if they understand that the two operations are inversely related to one another. Piaget was building on his idea of how children acquire conservation—that a property or object remains fundamentally unchanged despite some irrelevant transformation. In the classic conservation of liquid task, Piaget showed children two short and wide glasses both filled with the same amount of liquid and then poured the contents of one of them into a tall narrow glass. Piaget noted that a child who understood conservation could explain why the amount of liquid in both final containers—one short and
wide and the other tall and narrow—was the same, even though the glasses containing the same amounts of liquid were themselves different, by using the inversion explanation. If the liquid in the new tall and thin container was poured back into its original short and wide container then this would be the reverse or inverse of the original action and the amount of liquid would be demonstrably the same. Piaget applied this notion of conservation or quantitative identity and how an amount stayed unchanged despite an irrelevant physical transformation to several different types of tasks, such as conservation of mass and conservation of length. He also extended it into his examination of children's concepts about number. His work about how children understand mathematics is considered integral for educators and researchers concerned with how children's performance in mathematics is hindered when an understanding of mathematics is not an integral part of their learning (Fuson, 2009).

Using Piaget's ideas, Starkey and Gelman (1982) tested whether children understood the concept of inversion as it pertained to addition and subtraction by asking 3-, 4-, and 5-year-olds to solve problems, such as $1 + 2 - 2$ using concrete objects. If children understood that addition and subtraction are inversely related then they would not need to perform any calculations to arrive at the correct answer of 1 as the same number was both added and subtracted. Starkey and Gelman found children's accuracy on these problems was moderate to high across development and hypothesized that solving the problems accurately could mean that a child understood the concept of inversion as it applies to the operations of addition and subtraction. Although not without some methodological issues, this study has served as an important cornerstone for subsequent research on children's understanding of arithmetic concepts.

**WHAT IS CONCEPTUAL KNOWLEDGE OF ARITHMETIC?**

Over the years, there have been a number of definitions proposed for conceptual knowledge of arithmetic and therein lies one of the difficulties of conducting research in this area (Crooks & Alibali, 2014). A second difficulty is finding measures of knowledge—a point that will be discussed later. With respect to the first point, the issue is that if there is no consensus about the definition of conceptual knowledge, then how can studies of conceptual knowledge from different labs using different measures or tasks be compared?

A number of definitions have been proposed over the last 30 years or so. Hiebert and Lefevre (1986) defined conceptual knowledge as an interconnected web of knowledge that includes how information within a domain, such as mathematics is related. Kilpatrick et al. (2001) proposed that conceptual knowledge is the understanding of mathematical concepts, operations, and relations. Similarly, Baroody and Ginsburg (1986) considered conceptual knowledge to be implicit or explicit understanding of concepts or principles. This definition has been used by a large number of researchers across a wide variety of mathematical concepts (e.g., Rittle-Johnson & Alibali, 1999). Bisanz and LeFevre (1990) further elaborated on this definition by noting that conceptual knowledge reflects the underlying cognitive processes involved in understanding, and how to apply (p. 216).

Although still a contentious area, conceptual knowledge should be an important consideration that will provide a more coherent and variety of definitions of conceptual knowledge discussed in this chapter. An important aspect of conceptual knowledge is how it is identified by Crooks and Alibali (2014).

**THE IMPORTANCE OF CONCEPTUAL KNOWLEDGE IN MATHEMATICS LEARNING**

Despite the lack of a consensus for conceptual knowledge, there does appear to be a general agreement among researchers that a strong foundation in early childhood should be an important component of mathematics education. The National Mathematics Advisory Panel (2008) emphasized that understanding is not enough; children need to be able to appropriately translate abstract mathematical concepts into understandings novel and meaningful for them. This understanding can also be an important factor in the development of procedures (Baroody & Alibali, 2001). Thus, the idea that the acquisition of concepts can be enhanced by the use of concrete objects (e.g., Hanich, Heth, & Ak给, 2001). Therefore, an accurate answer to the question of how children learn mathematics.

Even though it is difficult to know the extent to which important concepts are learned, it appears that future research in mathematics education should focus on how children learn these concepts. Bisanz (1999) noted that the definition of conceptual knowledge and the way it is used can vary. Consequently, although there is no consensus on what constitutes conceptual knowledge, more attention has been given to the importance of knowing how children learn about these concepts.

**A BRIEF HISTORY OF RESEARCH ON CONCEPTUAL KNOWLEDGE**

Research on conceptual knowledge has been directed towards understanding how children learn mathematical concepts and operations. This research has been conducted within a variety of educational settings, including elementary and secondary schools, as well as in informal learning environments. The literature on conceptual knowledge of arithmetic has been characterized by a focus on the development of children's understanding of mathematical concepts and operations. This focus has been driven by an interest in understanding how children learn mathematics and how to improve children's mathematical performance. The research on conceptual knowledge of arithmetic has been characterized by a focus on the development of children's understanding of mathematical concepts and operations. This focus has been driven by an interest in understanding how children learn mathematics and how to improve children's mathematical performance. The research on conceptual knowledge of arithmetic has been characterized by a focus on the development of children's understanding of mathematical concepts and operations. This focus has been driven by an interest in understanding how children learn mathematics and how to improve children's mathematical performance.
was the same, even though the glasses were themselves different, by using the new tall and thin container. Then this would be the reverse or amount of liquid would be demonstrably conservation or quantitative identity and create an irrelevant physical transformation as conservation of mass and conservation. His examination of children’s concepts children understand mathematics is concerned with how children’s when an understanding of mathematics (Fuson, 2009).

Selman (1982) tested whether children it pertained to addition and subtraction the problems, such as 1 + 2 = 2 using concept addition and subtraction are inversely perform any calculations to arrive at the answer was both added and subtracted. Star- tency on these problems was moderate to realized that solving the problems accurately the concept of inversion as it applies to addition. Although not without some method- an important cornerstone for subsequent arithmetic concepts.

DETONATION OF ARITHMETIC?

Number of definitions proposed for conceptual critical for children’s ability to identify and correct errors, for appropriately transferring algorithms to solve novel problems, and for understanding novel problems in general (pp.4–14).” Conceptual knowledge may also be an important basis for how children discover new problem solving procedures (Baroody, 2003; Rittle-Johnson, Schneider, & Star, 2015). Knowledge of concepts can provide shortcuts during problem solving procedures (Jordan, Hanich, & Uberti, 2003) and therefore result in faster problem solving and more accurate answers (Lai, Baroody, & Johnson, 2008).

Even though there is an almost universal acknowledgement about how important conceptual knowledge of mathematics is—not only for current but future mathematics learning—research on this knowledge has lagged far behind research on the other two types of knowledge (procedural and factual). Bisanz (1999) noted that the role of conceptual understanding of how children’s knowledge and skills in mathematics develop was underresearched. Interestingly, although the development of children’s conceptual knowledge is attracting more attention from researchers (Crooks & Alibali, 2014), much remains to be learned about this development (Gilmore & Papadatou-Pastou, 2009).

A BRIEF HISTORY OF RESEARCH ON CONCEPTUAL KNOWLEDGE

Research on conceptual knowledge has been hindered not only by definitional ambiguity, but also by how to best measure this knowledge (Prather & Alibali, 2009). Most researchers have concluded conceptual knowledge is best measured indirectly (Bisanz & LeFevre, 1990; Canobi, 2009; Carpenter, 1986;
The most commonly used method is to present children with arithmetic problems that are most easily and quickly solved if children have knowledge of the underlying concepts, principles, or relations. For example, if children understand that addition and subtraction are inversely related operations, even when presented with a problem, such as $354297 + 8638298 - 8638298$, they should be able to quickly and accurately solve the problem by stating the first number. This approach is typically called the inversion shortcut (Bisanz & LeFevre, 1990). If, however, they do not understand that addition and subtraction are inversely related and they do not have a calculator or even pencil and paper handy, this problem will take most children a long time to solve and they are likely to make a calculation error.

Simply asking children what they know, for example, about the relation between addition and subtraction often does not yield information relating to children’s knowledge of the inverse relation between addition and subtraction. Instead, researchers have used novel tasks, such as the inversion problem (Starkey & Gelman, 1982). When students encounter new or novel problems, they can make use of their conceptual knowledge to generate novel problem solving procedures (Schneider, Rittle-Johnson, & Star, 2011). The generation of these procedures suggests that the students at least implicitly understand the associated concepts. If children spontaneously report using conceptual knowledge to solve problems then they are demonstrating explicit knowledge which is typically considered a more stringent measure of their conceptual understanding (Canobi, 2009) as compared to, when they are taught conceptually-based problem solving procedures (Carpenter, 1986).

So, when measuring conceptual knowledge through the use of problem solving procedures—a notion integral in Bisanz and LeFevre’s (1990) definition of conceptual knowledge, the use of novel problems is important. Novel problems mean that children must spontaneously generate a new problem solving procedure or transfer a known procedure from a conceptually similar but superficially different problem. In this way, there is no possibility that children are using the rote application of a previously learned procedure. Application of such a rote learned procedure would mean that children are not required to understand the concepts or principles being assessed in order to solve the problem successfully. Moreover, the necessity of understanding a specific concept or principle in order to generate a problem solving procedure yields important information about the strategies that children can spontaneously use (Dixon & Moore, 1996) so it helps researchers and educators to not only know more about children’s conceptual knowledge of arithmetic and, consequently also provides insights into children’s procedural knowledge.

It so happens that, for most children, the inversion problems originally formulated by Starkey and Gelman (1982) are novel to them. In school, particularly in the early years, they spend most of their time dealing only with two-term problems (e.g., $3 \times 4$) or three-term problems involving the same operations (e.g., $2 + 5 + 13$). Three-term (or more) problems only become more common in the late elementary and middle school years. For some problems, researchers have also used the inversion task, not only as a measure of children’s understanding, but also as a means to verify that is integral to solving the problem (e.g., Starkey & Gelman, 1982). If, however, they do not understand that addition and subtraction are inversely related problems, they will not be able to solve them accurately (Gelman, 1975). Thus, it is clear that the key role in these tasks is to ensure that children are able to use their conceptual understanding to solve the problem.

Bisanz and LeFevre’s (1990) definition of conceptual knowledge include not only the ability to use conceptual knowledge to solve problems but also the ability to generate a new problem solving procedure. If children spontaneously report using conceptual knowledge to solve problems then they are demonstrating explicit knowledge which is typically considered a more stringent measure of their conceptual understanding as compared to, when they are taught conceptually-based problem solving procedures (Carpenter, 1986).

Second, Ashcraft (1988) and Ashcraft and Faust (1989) also found that when solving addition problems, children are slower and make more errors if they have been taught to solve the problem accurately by rote practice. Then, if children are taught the problem, such as $2 + 5 + 13$, and included in their teaching is the fact that the problem is required to be solved using rote practice, they were solved accurately and quickly. However, they found that children’s procedural knowledge is the same whereas their conceptual knowledge increased, suggesting that the rote methodological approach is not always effective.
The commonly used method is to present the underlying concepts, principles, or relationships that addition and subtraction are inversely related and they do not have an underlying concept or principle. Researchers have proposed that understanding of the inverse relation between addition and subtraction is integral for learning algebra (Nunes et al., 2008). In this way, inversion problems (e.g., \(2 + 23 - 23\)) are ideal for assessing children’s understanding of the inverse relation between addition and subtraction and multiplication and division (Gilmore & Papadatou-Pastou, 2009). Bisanz and LeFevre (1990) played a key role in taking Starkey and Gelman’s inversion problems and developing a design to effectively assess conceptual knowledge.

Bisanz and LeFevre (1990) not only assessed children’s accuracy on inversion problems (e.g., \(6 + 3 - 3 = \_\)), they also asked children to provide a verbal report of their problem solving procedure. In this way, Bisanz and LeFevre could differentiate between children who were calculating the answer by adding the first two numbers (e.g., \(6 + 3\)) and then subtracting the third number (e.g., \(9 - 3\)) from those children who were applying their understanding of the inversion concept. That is, the children reported that they did not add or subtract because adding and subtracting left the first number, \(6\), unaffected—they applied the conceptually-based inversion shortcut. Additionally, Bisanz and LeFevre included a second type of problem, which they called the standard problem, as a further check for whether children applied their conceptual knowledge of the inverse relation between addition and subtraction. While inversion problems are of the form \(a + b - b\), standard problems are of the form \(a + b - c\) (e.g., \(6 + 5 - 2\)). The use of standard problems cleverly served two purposes. First, on standard problems, unlike inversion problems, calculation is required. Therefore, slower problem solving and more errors should occur and children should verbally report using a calculation strategy on standard problems than when using the inversion shortcut on inversion problems.

Second, a problem size effect can also be assessed with standard problems. Ashcraft (1982) first investigated the problem size effect on two-term problems and found that larger numbers in a problem were associated with more errors and slower problem solving. A problem, such as \(2 + 4\) would tend to be solved accurately, and relatively quickly, even by young children, compared to a problem, such as \(265 + 382\) or even \(8 + 9\). By manipulating the size of the numbers included in both the inversion and standard problems, Bisanz and LeFevre posited that the problem size effect would be found for standard problems that required calculations but would not be found for inversion problems if children were solving these based on their conceptual knowledge. This is the pattern they found for 6-, 9-, and 11-year-olds as well as adults. On inversion problems, children’s accuracy did not vary with the size of the numbers in the problem, whereas on standard problems, as the size of the numbers in the problem increased, so did the errors. With their innovative inclusion of the standard problem and assessment of the problem size effect, Bisanz and LeFevre laid out the methodological groundwork for all subsequent work on the inversion concept.
Overall, the bulk of research on conceptual knowledge has tended to focus on a fairly restricted set of concepts. As Crooks and Alibali (2014) note, equivalence and inversion are two of the most thoroughly researched arithmetic concepts. Equivalence is the focus of the chapter by McNeil and coworkers in Chapter 8 so it will not be considered any further here. Other concepts, such as commutativity (e.g., Baroody, Ginsburg, & Waxman, 1983; Canobi, 2005; Cowan & Renton, 1996), that is if \( a + b = c \) then \( b + c = a \), have been investigated, but as they have not received as much research attention it is more difficult to draw strong conclusions from them compared to the concepts of inversion and equivalence (Crooks & Alibali, 2014; Gilmore & Papadatou-Pastou, 2009; Prather & Alibali, 2009). Also, concepts, such as commutativity are usually explicitly taught to children (Canobi & Bethune, 2008) so, unlike novel problems, such as inversion and equivalence problems, it is not clear whether children are applying their conceptual knowledge when solving these problems or applying a procedure that they were taught and the conceptual basis of which they may not understand (Cowan, 2003).

THE IMPORTANCE OF MULTIPLICATIVE CONCEPTS AND THE STATE OF CURRENT RESEARCH

What is notable about most of the research on arithmetic concepts is that almost all researchers have focused exclusively on additive concepts, that is, concepts involving the operations of addition and/or subtraction. Research on children’s understanding of multiplicative concepts, that is, concepts involving the operations of multiplication and/or division, is relatively sparse, despite its importance for more complex and advanced mathematical skills (Cowan & Renton, 1996; NMAP, 2008; Nunes et al., 2009). This is potentially problematic for a number of reasons. First, multiplication and division are typically considered to be more difficult operations for children to understand and master, particularly division (Dixon, Deets, & Bangert, 2001). As a result, concepts about the relationship between multiplication and division may also be more difficult than concepts about the relationship between addition and subtraction (Fuson, 1988). This leads to the second reason; specifically, just because a child understands a concept on an additive problem (e.g., \( 2 + 3 + 8 = 2 + 11 \)), it does not mean that they will understand the same concept on a multiplicative problem (e.g., \( 2 \times 3 \times 8 = 2 \times 24 \)) (Nunes, Bryant, Hallett, Bell, & Evans, 2009). Third, there is evidence that children may develop an implicit understanding of the inverse relation between addition and subtraction (Klein & Bisanz, 2000), but coming to understand the inverse relation between multiplication and division is likely to require formal instruction.

ADDITIVE VERSUS MULTIPLICATIVE CONCEPTS

The two arithmetic concepts of inversion and associativity have been investigated using both additive and multiplicative problems. Both concepts involve understanding the relations among arithmetic operations—an understanding that is critical to the development of mathematical reasoning.
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research attention it is more difficult

performed on three measures is recom-

veridical.

THE INVERSION CONCEPT

The inversion concept has been studied extensively from the preschool years
to adulthood using addition and subtraction problems of the form

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Research on children’s knowledge of inversion

and associativity have been investi-

Both concepts involve

multiplicative operations—an understanding

that numbers can be, for example, added in any order and the result will be the same (e.g., 3 + 4 + 5 can be solved correctly by adding 3 and 4 and then adding 5 or by adding 4 and 5 together and then adding 3). Most of the research on associativity has focused on one operation at a time and, typically, the operation of addition or subtraction (e.g., Shumway, 1974), but associativity can also be applied to the standard problems devised by Bisanz and LeFevre (1990). These standard problems can also be used to assess conceptual knowledge so are often called associativity problems.
(Klein & Bisanz, 2000). Associativity problems are of the form $a + b - c$ and $d \times e \div f$ (e.g., $4 + 9 - 3$ and $4 \times 9 \div 3$) and can often be more quickly and accurately solved if the second and third numbers are dealt with first (Klein & Bisanz, 2000; Robinson & Ninowski, 2003) (i.e., $9 - 3 = 6 + 4$ and $9 \div 3 = 3 \times 4$). The advantage of using the associativity shortcut becomes obvious on problems, such as $453 + 987 - 972$ or $23 \times 39 = 13$. If these problems have to be solved using mental calculation, it is much faster and would result in fewer errors if $972$ was subtracted from $987$ before the addition of $453$ and if $39$ was divided by $13$ before multiplying by $23$.

There are two problem characteristics that can discourage the application of the associativity concept via the associativity shortcut. First, unlike inversion problems there is no clear visual pattern (i.e., $b - b = e$) that occurs in these associativity problems. Thus, applying the associativity concept requires more reflection on the part of the child in order to notice that there is an easier way to solve the problem. Second, some associativity problems are more conducive to the use of the associativity shortcut than others. On a problem, such as $6 + 23 - 21$, subtracting the third number from the second results in a positive number whereas, on a problem, such as $7 + 25 - 29$, subtracting the third number from the second results in a negative number. Negative numbers are more challenging for children to understand and use during problem solving (Young & Booth, 2015) and therefore may prevent children from applying their conceptual knowledge of associativity even if they do notice the associativity shortcut can be used. This issue is further compounded on multiplicative problems. Using the associativity shortcut to solve $6 \times 18 = 2$ is much more straightforward than using the shortcut to solve $4 \times 9 = 27$; this is because the latter would involve a fraction and children are often not comfortable dealing with even simple fractions (see also Van Hoof et al., Chapter 5; as well as Hallett, Nunes, Bryant, & Thorpe, 2012; Siegler & Lortie-Forgues, 2015). Finally, using the associativity shortcut requires calculation, just as the left-to-right problem solving procedure does. If problem solvers are seeking to use the most efficient problem solving procedures (Siegler & Araya, 2005) then some students may not realize or deem the associativity shortcut to be sufficiently more efficacious than the left-to-right approach.

**ARE ADDITIVE AND MULTIPLICATIVE CONCEPTS THE SAME?**

Robinson and LeFevre (2012) explored the issue of whether children understand additive and multiplicative concepts, specifically the concepts of inversion and associativity, in the same way. If this is the case, then children who understand the additive version of the concept should also understand the multiplicative version. Nunes et al. (2008) pointed out that, for at least the concept of inversion, understanding the relation between addition and subtraction should be a precursor for understanding the relation between multiplication and division and this point can be made quite well. Enough studies have been done on both types of problems by researchers to support the idea that both types of problems can be used to help understand the concept of inversion. The question remains as to whether understanding the concept of inversion is the same for both types of problems. Robinson and Ninowski (2003) have argued that it is not. They have suggested that understanding the concept of inversion is different for addition and subtraction inversion problems.

**Inversion**

Formal schooling does not teach the inversion concept on addition and subtraction (Klein & Bisanz, 2000; Rasmussen, 2003). Early understanding, concepts, and symbolic versions of inversion problems are understood about a third of problems increasing to almost two-thirds on almost all problems (Robinson & Ninowski, 2003).

The inversion shortcut on addition problems. Overall, less than a quarter of the multiplicative problems and Robinson & Ninowski, 2003). Dubé (2014), but only a third of Robinson & Ninowski, 2003) that emerges: both children on multiplication and division.

Two studies provide further knowledge of inversion concept. Siegler and Stermer (2012) and applied the inversion shortcut on their study, over a third of their study, on almost all problems (Robinson & Ninowski, 2003).

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problems are of the form \(a + b - c\) and can often be more quickly calculated than \(9 \div 3\) and third numbers are dealt with first (Klein, Bisanz, 2000; Rasmussen et al., 2003) (i.e., \(9 - 3 = 6 + 4\) and the associativity shortcut becomes obvious for some). If these problems are dealt with first, it is much faster and would result in 67 instead of 13. However, studies have been conducted that either directly compare additive and multiplicative versions of inversion and associativity or have used similar research designs to address these issues.

**Inversion**

Formal schooling does not seem to be required for children to understand the inversion concept on addition and subtraction inversion problems (Klein & Bisanz, 2000; Rasmussen et al., 2003). However, in most of these studies on early understanding, concrete objects (e.g., XX + XXXX − XXXX) rather than symbolic versions of inversion problems (e.g., \(2 + 4 - 4\)) have been used. When symbolic problems are used, however, the inversion concept is applied on only about a third of problems in 2nd grade (Robinson & Dubé, 2009a, 2013), increasing to almost two-thirds in 8th grade (Robinson et al., 2006), and then on almost all problems for undergraduate university students (Robinson & Ninowski, 2003).

The inversion shortcut is used less frequently on multiplication and division problems. Overall, 6th graders use the inversion shortcut on less than a quarter of the multiplication and division problems (Robinson & Dubé, 2009b; Robinson & Ninowski, 2003), increasing to about half the problems in 9th grade (Dubé, 2014), but over two-thirds of problems in undergraduates (Dubé, 2014; Robinson & Ninowski, 2003). Across a series of studies there is a clear pattern that emerges: both children and adults are less likely to use inversion shortcuts on multiplication and division inversion problems.

Two studies provide further insights into the difference in conceptual knowledge of inversion between additive and multiplicative versions of the concept. Siegler and Stern (1998) conducted a microgenetic study in which 2nd graders who did not spontaneously use the inversion shortcut on addition and subtraction problems were presented with a set of inversion problems once a week for several weeks. By the end of the study, every student had discovered and applied the inversion shortcut during problem solving. Robinson and Dubé (2009c) used the same design with 6th graders who did not spontaneously use the inversion shortcut on multiplication and division problems. By the end of their study, over a third of participants had still not discovered and applied the inversion shortcut. It appears that it is easier for children to induce the inverse relation between addition and subtraction than that between multiplication and division.

Only one study with children has directly compared performance on addition and subtraction inversion problems with performance on multiplication and division inversion problems. Robinson et al. (2006) had 6th and 8th graders solve both types of problems but half of the children solved the addition and subtraction problems first and the other half the multiplication and division problems first. Not surprisingly, inversion shortcut use was higher on the addition and
subtraction problems. The key findings related to Robinson et al.’s hypothesis that because knowledge of the additive inversion concept develops even before formal schooling, solving the addition and subtraction problems first might increase inversion shortcut use on the multiplication and division problems. However, contrary to this hypothesis, no transfer effects were found. Robinson and Ninowski (2003) found a similar lack of transfer with undergraduates. The students who solved the addition and subtraction inversion problems first were no more likely to subsequently use the inversion shortcut on the multiplication and division inversion problems than those who had solved the multiplication and division problems first. This is consistent with the studies described earlier and provides support for the implication that there is a disconnect between the additive and multiplicative versions of the inversion concept. The results of these studies as well as the two microgenetic studies provide further evidence that children and adults have a relatively weaker understanding of the inversion concept when it is being assessed with the operations of multiplication and division compared to the operations of addition and subtraction.

**Associativity**

Klein and Bisanz (2000), in a study of 4-year-olds’ solving of concrete problems, were the first to notice that standard problems of the form \(a + b - c\) can be solved using conceptual knowledge. They found that although almost a third of 4-year-olds used the associativity shortcut at least once, overall use was only 5%. Other research has shown that second graders’ use of the associativity shortcut ranges from 10% to just under 15% of the problems, from 10% to 30% in 3rd grade, from under 10% to just over 25% in 4th grade, and to 25% by 5th grade (Robinson & Dubé, 2009a, 2012, 2013), reflecting a tendency for the number of students who use the shortcut to increase across grade rather than more students using it occasionally as they get older. One study has examined associativity shortcut use on addition and subtraction problems (in this study called the “right-to-left” strategy) in older students with about 10% use in 6th grade and less than 25% in 8th grade (Robinson et al., 2006). The difference between these studies and studies with higher rates of shortcut use may be due to the inclusion in this study of standard problems that were not conducive to associativity shortcut use. For some of the problems in Robinson et al. (2006), subtracting the third from the second number resulted in a negative number; so this study may underestimate children’s understanding of the associativity concept on addition and subtraction problems.

Overall, however, associativity shortcut use seems to develop very slowly across childhood. Even undergraduates only use the shortcut to solve about 60% of the problems on which it could be used (Robinson & Ninowski, 2003), but once again this study included problems that were not as conducive to use of the associativity shortcut. To examine whether associativity shortcut use is impacted by problem format, LeFevre and Robinson (2015, unpublished manuscript) separated the standard problems into problems that were “conducive” to associativity shortcut use and problems that were not conducive” (e.g., \(9 + 25 - 17\)). They found that about 16% of the problems were conducive and 54% were not conducive, suggesting that problem format influences shortcut use.

On multiplication and division problems, it may be even further. Robinson et al. (2006) found that second graders used the shortcut, even though they used it less often than first graders. Robinson and Ninowski (2003) found a similar lack of transfer with undergraduates. The even further. Robinson et al. (2006) found that second graders used the shortcut, even though they used it less often than first graders. Robinson et al. (2003) also found a lack of transfer with undergraduates. The problem contains either an integer or a fraction when the associativity shortcut is available.
related to Robinson et al.’s hypothesis that the inversion concept develops even before addition and subtraction problems first might be posed. Moreover, transfer effects were found. Robinson and Ninowski (2003) found that in undergraduates, the substitution inversion problems first were used on problems that were “conducive” to using the shortcut, even on problems that were more conducive to it, and 8th graders used the shortcut on less than 5% of problems. More recently, Dubé (2014) used only conducive problems and found that associativity shortcut use ranged from under 15% in 7th grade, to over 25% in 9th grade, and to almost 40% in 11th grade. When presented with only conducive problems, undergraduate students’ use of the associativity shortcut ranged from 45% to 55% (Dubé, 2014; Dubé & Robinson, 2010a). Robinson and Ninowski (2003) used a mix of conducive and nonconducive problems and found that combined associativity shortcut use was 30%. However, LeFevre and Robinson (2015, unpublished manuscript) found that associativity shortcut use was 30% on nonconducive problems but rose to more than 75% on conducive problems that may be explained by the study’s inclusion of a large proportion of university students with very strong mathematical skills.

To sum up, the use of the associativity shortcut starts low and remains low across childhood and even among university students it is not the predominant strategy, particularly when the provisional result is a negative number or a fraction. Moreover, as with the inversion shortcut, associativity shortcut use is less frequent on multiplication and division problems compared to addition and subtraction problems.

Inversion Versus Associativity

A strong pattern emerges from the studies reported previously: students use the inversion shortcut more frequently than the associativity shortcut. This suggests that conceptual knowledge of inversion is stronger than that of associativity; but it is also possible that students are simply less likely to apply their knowledge of associativity than their knowledge of inversion through the use of shortcuts.

The first piece of evidence supporting the idea that students do have conceptual knowledge of associativity but are not always willing to apply it comes from the work of LeFevre and Robinson (2015, unpublished manuscript). As previously mentioned, they found that, on associativity problems that yielded a negative number or a fraction when dealing with the second and third numbers, associativity shortcut use dropped significantly on both additive and multiplicative problems. The second piece of evidence comes from the format of the problems themselves. On inversion problems there is always an obvious visual pattern; every inversion shortcut use is impact-
The third piece of evidence comes from the performance advantage of the inversion shortcut compared to the associativity shortcut. For the inversion shortcut, no calculations are required therefore very few errors occur on these problems (if they do it is usually because the participant states the b term by mistake) and problem solving solution times are very quick. In contrast, the associativity shortcut still requires calculation and therefore more errors and longer solution times will occur. For example, in a study by Robinson and Dube (2009a) found that the 46% use of the left-to-right problem solving strategy is associated with lower accuracy (90.2% vs. 44.7%) and shorter solution times (1818 vs. 2848 ms) when the inversion shortcut was used. When the associativity shortcut was used, overall accuracy (68.3% vs. 48.5%) and solution times (13,194 vs. 17,656 ms) were still better for the shortcut as compared to when a left-to-right calculation strategy was used, but the gains were smaller than for inversion problems. Robinson, Dube, and Beatch (2015) found a similar pattern for multiplication and division functions with 6th-8th graders. When the inversion shortcut was used, overall accuracy (90.2% vs. 44.7%) and solution times (1818 vs. 2848 ms) were better than when a left-to-right strategy was used. When the associativity shortcut was used, overall accuracy (51.4% vs. 36.4%) and solution times (2469 vs. 2993 ms) were better than when a left-to-right strategy was used. In particular, note that the solution time advantage of the associativity shortcut compared to using a left-to-right strategy is even smaller on multiplication and division inversion problems than on addition and subtraction problems (0.5 vs. 4.5 s, respectively) which may provide another reason why associativity shortcut use is particularly infrequent on multiplication and division problems—the performance advantage may not be a strong enough incentive to abandon well-learned and practiced left-to-right approaches to problem solving (Lemaire & Lecacheur, 2011; McNeil, Rittle-Johnson, Hattikudur, & Peterson, 2010).

There is also evidence that students’ conceptual knowledge of associativity is actually weaker than that of inversion. In most studies of the inversion and associativity concept, researchers have relied on the use of shortcuts to infer conceptual knowledge. However, other measures can be used (Bisanz et al., 2009; Prather & Alibali, 2009). One approach is to provide participants with a demonstration of the shortcuts and ask them to evaluate them. If children understand the concept behind the shortcuts they should provide positive evaluations of the shortcut and explanations for why the shortcuts are a good way to solve the inversion or associativity problems (Bisanz & LeFevre, 1992; Bisanz et al., 2009). This task is most commonly referred to as the “evaluation of procedures task” and has been used to investigate a range of mathematical concepts and tap into conceptual knowledge that children may not have been able to explicitly use or verbalize during problem solving (Crooks & Alibali, 2014).

In studies using both the problem solving task and the evaluation of procedures task, an interesting pattern of results has emerged and suggests that
from the performance advantage of the associativity shortcut. For the inversion shortcut there are very few errors occur on these problems. The participant states the $b$ term by left to right and longer times are very quick. In contrast, the associativity shortcut was used in a study of addition and subtraction problems. The inversion shortcut was used on inversion problems (57.4 vs. 55.4%) and solution times (6044 vs. 5586 ms) left-to-right calculation strategy was used when the associativity shortcut was used (90.2% vs. 44.7%) and solution times (2469 vs. 2993 ms) when a left-to-right strategy was used. When the associativity shortcut was used for accuracy (68.3% vs. 48.5%) and solution times there is greater advantage of the associativity shortcut over the inversion shortcut. In most studies of the inversion shortcut, the benefits of using the shortcut are not transferable to the associativity concept, and the benefits are not as large for multiplication and division problems compared to addition and subtraction problems (0.5 vs. 2.0). One reason why associativity shortcuts are used on multiplication and division problems—the strong enough incentive to abandon well-known strategies to problem solving (Lemaire & Innes, Hattikudur, & Peterson, 2010). Overall, the pattern is consistent regardless of which task is used to assess conceptual knowledge: children appear to have a better understanding of the inversion concept than the associativity concept. As a final note on this point, in all of the years that these studies have been conducted (not all of which are discussed in this chapter) and amongst the hundreds of participants involved, only one participant to date has used the associativity shortcut and not used the inversion shortcut. Every other associativity shortcut user, regardless of additive or multiplicative versions, has always used the inversion shortcut.

### Individual Differences and Factors in the Use of Conceptually-Based Shortcuts

To date, this discussion of arithmetic concepts has focused on the difference between additive and multiplicative concepts, between the concepts of inversion and associativity, and also outlined the pattern of developmental change in
these concepts but no mention has been made of individual differences and the factors that relate to conceptual knowledge; two areas of investigation which have recently been gaining interest (Gilmore & Papadatou-Pastou, 2009). Individual differences are considered first, followed by the examination of the factors relating to the use of conceptually-based shortcuts during problem solving.

INDIVIDUAL DIFFERENCES

Up until this point in the chapter, my focus has centered primarily on the use of the inversion shortcut during problem solving as a good measure of conceptual knowledge (Bisanz & LeFevre, 1990; Crooks & Alibali, 2014; Prather & Alibali, 2009). However, a more lenient measure can also be used. Bisanz and LeFevre (1990) noticed a third approach to solving inversion problems, which they termed the “negation strategy.” With this strategy, students use a mix of a left-to-right calculation procedure and the inversion shortcut. So, on a problem, such as $896 \times 5193 = 5193$, a participant might calculate the first part of the problem by multiplying 896 and 5193 together and then realize that dividing by the third number, 5193, negates the number they just multiplied leaving them the number they started with, 896. These participants have at least some understanding of the inversion concept because they understand the inverse or reverse relationship between multiplying and dividing (or adding and subtracting) the second and third numbers. In Siegler and Stern’s (1998) microgenetic study of the development of the inversion shortcut on addition and subtraction problems, they found that many students moved from using a left-to-right problem solving procedure, to using the negation strategy, to using the inversion shortcut. On multiplication and division inversion problems, the same pattern was found for some individuals but other individuals switched directly from using a left-to-right procedure to the inversion shortcut (Robinson & Dubé, 2009c).

Gilmore and Bryant (2006) were the first to use cluster analyses as a means to investigate individual differences in children’s understanding of arithmetic concepts. Cluster analysis is a statistical technique that sorts individuals into subgroups based on their performance on a range of tasks or measures. Robinson and Dubé (2009a) used cluster analyses to investigate individual differences in children’s understanding of both inversion and associativity on additive problems. In this instance, they only examined the use of the inversion and associativity shortcuts as measures of conceptual knowledge. They found three distinct clusters. The Dual Concept cluster, the smallest cluster included only 18% of the students who used both the inversion and associativity shortcuts frequently (87 and 80%, respectively). The Inversion Concept cluster included 40% of the students who used only the inversion shortcut frequently (69 and 16%). The No Concept cluster included 42% of the students who rarely used either the inversion or the associativity shortcuts (4 and 1%). Interestingly, grade did not predict cluster membership such that, for example, the 2nd graders were just as likely as the 3rd and 4th graders to be in the Dual Concept cluster.
made of individual differences and the second, two areas of investigation which focus has centered primarily on the use of inversion as a good measure of conceptual understanding (Crooks & Alibali, 2014; Prather & LeFevre, 2012). That is, the No Concept cluster (which is the Dual Concept cluster. (Robinson & LeFevre, 2012). That is, the No Concept cluster (which is

In a follow up study, Robinson and Dubé (2013) investigated individual differences in more detail with 3rd, 4th, and 5th graders by adding negation strategy use to the two shortcuts in their cluster analysis and a 4th cluster was identified: the Negation cluster. Once again, students in the Dual Concept cluster (16% of the participants) had frequent use of the inversion and associativity shortcuts and fairly infrequent use of the negation strategy (84, 74, and 4%, respectively). Students in the Inversion Concept cluster (19% of the participants) only used the inversion shortcut frequently and used the associativity shortcut and the negation strategy infrequently (70, 4, and 14%). Students in the Negation cluster (28% of the participants) rarely used the inversion or associativity shortcuts but frequently used the negation strategy (17, 6, and 56%). Finally, students in the No Concept cluster showed little evidence of conceptual understanding (3, 2, and 11%). Once again, grade did not predict cluster membership suggesting that the development of both the inversion and associativity concepts may not be as influenced by years of education as analysis considering only frequency of shortcut use might suggest.

For multiplication and division problems, a more detailed investigation of individual differences has been conducted. Robinson and Dubé (2015) drew on the data from several studies to conduct a cluster analysis based on the data from 540 participants in Grades 6, 7, and 8. This time, five clusters were identified. Students in the No Concept cluster (45% of participants) used the inversion and associativity shortcuts as well as the negation strategy infrequently (4, 2, and 7%, respectively). Students in the Negation Concept cluster (27% of participants) only used the negation strategy frequently (6, 1, and 59% for inversion, associativity, and negation, respectively), whereas, students in the Inversion Concept cluster (14% of participants) only used the inversion shortcut frequently (78, 3, and 11%). In this metaanalysis, the Dual Concept cluster identified in previous studies (both with additive and multiplicative problems) further subdivided into two further subgroups: the Low Dual Concept and the High Dual Concept clusters. These are both small clusters (4 and 10% for the Low and High Dual Concept clusters, respectively) which is presumably why they had not been identified as separate subgroups in previous studies. Students in the Low Dual Concept cluster used both shortcuts but only moderately so (34, 49, and 8%) whereas, students in the High Dual Concept clusters used both shortcuts frequently (89, 87, and 2%). Interestingly, the metaanalysis revealed that grade predicted cluster membership with 8th graders more likely to be in one of the Dual Concept cluster and for 7th graders more likely to be in the Negation cluster. This latter finding fits with the findings of Robinson and Dubé (2009b) who found that although overall inversion shortcut use and use of the negation strategy did not differ across grade in the problem solving task, 7th graders had the lowest preference for the shortcut compared to a left-to-right solution procedure (39% vs. 56% and 59% for 6th and 8th graders, respectively).

The original three clusters have also been identified in studies of adults (Robinson & LeFevre, 2012). That is, the No Concept cluster (which is
identified as the Weak Concept cluster because overall use of both shortcuts tends to be higher in all adults compared to almost all children), the Inversion Concept cluster, and the Dual Concept cluster were all found. That these three clusters have been found with younger children on additive problems, with older children on multiplicative problems, and with adults on both additive and multiplicative problems, suggests that these clusters represent patterns of individual differences that may be continuous across development.

FACTORS RELATING TO CONCEPTUALLY-BASED SHORTCUT USE

Computational Skills and Age
A number of researchers have been interested in the factors that predict the use of conceptually-based shortcuts. Most of the research has been with addition and subtraction inversion problems. Sherman and Bisanz (2007) looked at whether counting skills and knowledge of number names was related to performance on concrete inversion problems but only found weak correlations with arithmetic skills. Gilmore and Papadatou-Pastou (2009) conducted a metaanalysis of addition and subtraction inversion studies that included assessments of symbolic arithmetic skills and identified three clusters of individuals confirming previous research with the same clusters (Gilmore & Bryant, 2006). One cluster included children with both a strong understanding of the inversion concept and good calculation skills. A second cluster included children with a poor understanding of inversion and weak calculation skills. Finally, a third cluster included children with a good understanding of inversion but weak calculation skills. Interestingly, as has been found in other analysis of individual differences on the additive concepts, no grade differences were found in cluster membership and Gilmore and Papadatou-Pastou (2009) therefore concluded that although arithmetic skills generally increase across age or grade, the understanding of inversion does not. Watchorn et al. (2014) found a similar pattern of results with the same three clusters as well as a fourth cluster of participants with good arithmetic skills but low use of the inversion shortcut.

Only one study using a slightly different approach has examined whether computational skills are related to inversion shortcut use on multiplication and division problems and no research at all has been conducted for use of the associativity shortcut. Robinson and Dubé (2009b) identified three clusters of students comprised of 6th–8th graders. Students in the Inversion cluster used the inversion shortcut frequently (64%) and had the highest computational skills. Students in the Negation cluster used the negation strategy frequently (58%) and the inversion shortcut infrequently (5%) and their computational skills did not differ from students in the latter Computation cluster. These students rarely used the inversion shortcut (1%) or the negation strategy (3%) and instead relied on a left-to-right computational approach to solve the inversion problems.

Overall, it appears that these factors—individuals—to have a large impact on strategies. This suggests that further research is needed for the large insight.
because overall use of both shortcuts (to almost all children), the Inversion cluster were all found. That these same younger children on additive problems, problems, and with adults on both additive and these clusters represent patterns of various across development.

**Domain-General Cognitive Abilities**

It is suggested in the factors that predict to the individuals—to the understanding of the inverse relation between pairs of operations. This suggests that more than computational skills or age/grade account for the large individual differences in the understanding of arithmetic concepts.

**Working Memory**

Domain-general cognitive abilities have also been explored as potential contributors to individual differences in conceptual knowledge. Rasmussen et al. (2003) investigated the role of working memory in their study of the inversion concept on addition and subtraction problems with preschool and 1st graders. Corsi span, a measure of visual-spatial working memory, correlated with preschoolers' but not 1st graders' use of the inversion shortcut. Digit span, a measure of the phonological loop component of working memory, did not correlate with inversion shortcut use either in preschool or 1st grade students. Dubé and Robinson (2010b) investigated working memory and inversion shortcut use on multiplication and division problems with 6th and 8th grade students. In both grades, two measures of working memory, including the backward digit span, correlated with inversion shortcut use. Based on their findings they proposed that working memory might play a role in the use of the inversion shortcut because it helps direct attention towards the right side of an inversion problem (i.e., to the $e = e$ part of the problem). Watchorn et al. (2014) in their study on addition and subtraction inversion problems found that children with greater attention skills were more likely to use the inversion shortcut, particularly if they also had strong computation skills.

**Inhibition and Attention**

Dubé and Robinson (2010b) also hypothesized that the inhibition component of working memory may be particularly important for use of the inversion shortcut. Siegler and Araya (2005) proposed that to apply the inversion shortcut (at least on addition and subtraction problems), children must not only pay attention to the right side of the problem but also inhibit their well-established tendency to solve problems from left-to-right, and instead direct attention to the right side of the problem. Robinson and Dubé (2013) investigated the role of inhibition in children's use of the inversion and associativity shortcuts on addition and subtraction problems. In this study, as previously reported, four clusters of individuals were found: the No Concept, Negation, Inversion Concept, and Dual Concept clusters. Children in the two clusters with the highest use of conceptually-based shortcuts, the Inversion and Dual Concept clusters, also scored highest on the Stop-Signal task, a standard measure of inhibitory abilities. This finding suggests that these participants were able to inhibit their tendency to rotely solve problems from left-to-right and thereby process all of the presented numbers before executing the shortcut strategy.
No research to date has been conducted on the role of working memory or attention on the use of shortcuts when solving multiplication and division problems. However, a recent study by Dubé (2014) sheds some light on Siegler and Araya’s (2005) proposal that applying the inversion shortcut requires the interruption of the tendency to solve problems from left-to-right. Using multiplication and division inversion and associativity problems, Dubé (2014) had 7th, 9th, and 11th grade students, as well as undergraduate students solve problems using a task that checked whether they had calculated the product of the first two numbers. Calculation of the first two numbers on an inversion problem would mean that they did not inhibit or interrupt the left-to-right procedure to use the inversion shortcut which requires focusing on the right side of the problem. The same holds for the associativity shortcut which also requires focusing on the right side of the problem by dividing the second number by the third and then multiplying the result by the first number. Dubé (2014) found compelling evidence that inversion and associativity shortcut users had indeed inhibited their tendency to use left-to-right problem solving procedures in order to implement their use of the shortcuts. These findings suggest that inhibition is likely to also be involved in the use of conceptually-based shortcuts on multiplication and division problems and for associativity problems as well as inversion problems, at least on problems typically used in these studies (cf., 43 − 43 + 78 or 64 ÷ 32 × 21). In sum, working memory, attention, and inhibition all need to be considered as important cognitive factors in the development of arithmetic concepts as well as in theories of mathematical development (Cragg & Gilmore, 2014).

Attitudes

Although less research has been conducted in this area, there is evidence that children’s attitudes can influence their use of conceptually-based shortcuts. In these instances, children may have the necessary conceptual knowledge but may refuse to use it during problem solving. This is potentially problematic given that researchers have long relied on the use of conceptually-based shortcuts as one of the best ways, if not the best way, to measure conceptual knowledge (Bisanz et al., 2009; Crooks & Alibali, 2014; Gilmore & Papadatou-Pastou, 2009; Prather & Alibali, 2009). In early studies on 3rd–8th graders and using the evaluation of procedures task (Robinson & Dubé, 2009a, 2009b), it became apparent that some children had strong attitudes as to whether the inversion and/or associativity shortcuts were “good” or appropriate problem solving procedures. Some students, when told about how a fictional child had used the inversion or associativity shortcut to solve a problem, reacted with strong approval and when asked for their reasoning often expressed admiration for the cleverness of the fictional child, noted how efficient or smart the shortcut was, and some even wished that they had thought of the shortcut themselves when they had solved the inversion and associativity problems earlier in the session. In contrast, other students questioned the ability of the fictional child of choosing the shortcut and how they relate to them, having helped fill this gap in solving two sets of addition problems. Between the two sets of the task in which the inversion or associativity problems were solved, the shortcut or the left-to-right procedure was used on the first set of problems. For the second set, higher for students in the group that increased significantly their use of the shortcut. This suggests two possible ways in which children’s understanding of the attitude towards the demonstration of the associativity shortcut is possible that more explicit instruction would influence the use. Although this would not be the case in the current study.

In a more recent study on 3rd-8th grade students’ attitudes towards problem solving, they experienced their use of the evaluation of procedures task (Robinson & Dubé, 2009a, 2009b). It became apparent that some children had strong attitudes as to whether the inversion and/or associativity shortcuts were “good” or appropriate problem solving procedures. Some students, when told about how a fictional child had used the inversion or associativity shortcut to solve a problem, reacted with strong approval and when asked for their reasoning often expressed admiration for the cleverness of the fictional child, noted how efficient or smart the shortcut was, and some even wished that they had thought of the shortcut themselves when they had solved the inversion and associativity problems earlier in the session.
In contrast, other students reacted with strong disapproval. Some students stated that when solving problems, no steps should be omitted or done out of order therefore the shortcuts were not appropriate, and some stated that when solving math problems calculations always had to be performed. Other students questioned the abilities of the fictional child’s teacher and others accused the fictional child of cheating. These positive and negative responses were found with children in all grades, for both inversion and associativity, and on both additive and multiplicative problems.

Kilpatrick et al. (2001) and others (e.g., Ellis, 1997; Martin, Anderson, Bobis, Vellar, & Way, 2012) have recognized how attitudes toward mathematics can impact success in mathematics. Researchers have found that noncognitive factors, such as mathematics anxiety negatively affect mathematics performance (e.g., Ashcraft & Rudig, 2012; Ramirez, Chang, Maloney, Levine, & Beilock, 2016) but little research has specifically examined attitudes involving procedures in order to implement shortcuts on multiplication and division problems as well as inversion and associativity shortcuts were demonstrated along with a left-to-right procedure and students were asked whether they preferred the shortcut or the left-to-right procedure. The students then solved the second set of problems. For both inversion and associativity, overall shortcut use was higher for students in the second set. In particular, associativity shortcut use increased significantly more for the participants who had preferred the shortcut. This suggests two possibilities: that the evaluation of procedures task promoted understanding of the associativity concept or that students felt that, because of the demonstration of the shortcut, it was appropriate for them to now start using the associativity shortcut. However, as there was no control group it is also possible that more exposure to the problems alone promoted shortcut discovery though this would not be consistent Robinson and Dubé’s (2009c) microgenetic study.

In a more recent study, Robinson et al. (2016) investigated how 6th–8th grade students’ attitudes about the inversion and associativity shortcuts influenced their use of the shortcuts during multiplication and division problem solving. Half of the students solved two sets of problems consecutively while the other half completed the evaluation of procedures task between the two sets of problems. On inversion problems, inversion shortcut use increased across problem set for both groups but the increase was greater for the group that completed the evaluation of procedures task between the problem sets. Conversely, on associativity problems, associativity shortcut use only increased across problem set for the group that completed the evaluation of procedures task between the problem sets. Preference for a shortcut versus a left-to-right procedure also influenced their use. For this, only data from the group that completed the evaluation of procedures task between the two problem sets was analyzed. For both
inversion and associativity problems, students with a preference for the inversion shortcut used it more frequently than other students. Therefore, there is evidence that the evaluation of the procedures task itself lead to an increase in conceptual knowledge or at least an increased willingness to apply conceptual knowledge, but there is also evidence that children's attitudes towards the demonstrated shortcuts in the evaluation of procedures task influenced their subsequent shortcut use. The impact of attitudes on conceptually-based shortcut use is concerning but is consistent with recent findings that mathematics anxiety prevents some children from using more advanced problem solving strategies (Ramirez et al., 2016). Overall, the current research on attitudes adds further complexity to the variables involved in the development of conceptual knowledge as assessed through the use of conceptually-based shortcuts.

Educational Experiences

One final factor needs to be briefly discussed. Contradictory evidence exists on the role of age or grade on the use of conceptually-based shortcuts as some studies have found that use of the inversion and/or associativity increases across grade while others have not (e.g., Gilmore & Papadatou-Pastou, 2009; Robinson & Dubé, 2015). By adulthood, inversion and associativity shortcut use are at their highest (e.g., Dubé, 2014), suggesting that conceptual knowledge has developed since childhood and adolescence, but within tighter time spans (e.g., from 2nd to 4th grade), often there is little conceptual change (e.g., the finding that grade often does not predict cluster membership). Therefore, instruction is probably important in the development of conceptual knowledge but for some students, this does not appear to be sufficient for them to understand the concepts of inversion and associativity. Other educational factors, however, may impact mathematical performance.

Harold Stevenson, in his examination of cross-cultural differences in academic achievement consistently found that American children underperformed in mathematics in comparison to East Asian children. Stevenson et al. (1990) attributed Americans’ underperformance to children’s lower motivation toward mathematics, parents’ lower expectations for their children in mathematics, teachers’ lower levels of interest in teaching mathematics, and a less rigorous mathematical curriculum. Laski and Yu (2014) concluded that one of the reasons that Chinese outperform Chinese-American Kindergarten children on arithmetic tasks was due to Chinese teachers spending more class time discussing and explaining numerical concepts. Ma (1999) has proposed that one of the most marked differences between Chinese and American teachers relates to their conceptual understanding of elementary mathematics. A number of researchers have examined cross-cultural differences in conceptual knowledge (Ho & Fuson, 1998; Miller, Major, Shu, & Zhang, 2000; Torbeyns, Schneider, Xin, & Siegler, 2015) but only one has focused on the concepts of inversion and associativity. Robinson and Beatch (2015) had Canadian-educated adults and adults who had previously lived in Asia (mostly in China) assess their understanding of the equivalent of the procedures task. In this study, adults who had previously lived in Asia (mostly in China) assessed procedures task influenced their subsequent shortcut use. Therefore, there is evidence that the evaluation of the procedures task itself lead to an increase in conceptual knowledge or at least an increased willingness to apply conceptual knowledge, but there is also evidence that children’s attitudes towards the demonstrated shortcuts in the evaluation of procedures task influenced their subsequent shortcut use. The impact of attitudes on conceptually-based shortcut use is concerning but is consistent with recent findings that mathematics anxiety prevents some children from using more advanced problem solving strategies (Ramirez et al., 2016). Overall, the current research on attitudes adds further complexity to the variables involved in the development of conceptual knowledge as assessed through the use of conceptually-based shortcuts.

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Conclusions and recommendations

The NMAP (2008) project found that conceptually-based shortcuts were likely important in the development of conceptual knowledge as assessed. As Robinson and Beatch (2015) noted, theories of how children acquire concepts, typically rely on instruction and do not include a consideration of educational factors that are involved in children's conceptual understanding.

The NMAP's (2008) report highlighted the need for further research on the role of age or grade on the use of conceptually-based shortcuts as some studies have found that use of the inversion and/or associativity increases across grade while others have not (e.g., Gilmore & Papadatou-Pastou, 2009; Robinson & Dubé, 2015). By adulthood, inversion and associativity shortcut use are at their highest (e.g., Dubé, 2014), suggesting that conceptual knowledge has developed since childhood and adolescence, but within tighter time spans (e.g., from 2nd to 4th grade), often there is little conceptual change (e.g., the finding that grade often does not predict cluster membership). Therefore, instruction is probably important in the development of conceptual knowledge but for some students, this does not appear to be sufficient for them to understand the concepts of inversion and associativity. Other educational factors, however, may impact mathematical performance.

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students with a preference for the inversion procedures task itself lead to an increase in conceptual knowledge of arithmetic concepts. Asian participants used more shortcuts overall and were more likely to be placed into the equivalent of the Dual Concept cluster discussed earlier. Findings from this study support the notion that cross-cultural differences exist in conceptual knowledge even in adulthood, mostly suggesting that early educational experiences have long term implications for how well the concepts of inversion and associativity are understood.

In summary, the growing body of research on both additive and multiplicative concepts demonstrates the complexity of factors relating to not only conceptual knowledge of arithmetic but also to how conceptual knowledge is assessed. As Robinson and LeFevre (2012) pointed out, current research, and theories of how children’s concepts develop are based primarily on additive concepts, typically rely on studies which focus on only one concept at a time, and do not include a broad range of measures to investigate the many factors that are involved in children’s developing conceptual knowledge of arithmetic.

CONCLUSIONS AND FUTURE DIRECTIONS

The NMAP (2008) proposed that knowledge of mathematical concepts is an important predictor of success in later mathematics learning. They took specific note of the inverse relation between addition and subtraction as well as the inverse relation between multiplication and division but they concluded that American children’s weak understanding of these and other core arithmetic concepts was not acceptable and highlighted the need for curricular change.

The NMAP’s (2008) singling out of a multiplicative concept as a specific area of concern highlights the need for researchers to increase their focus on children’s understanding of multiplication and division. These two operations and the relations between them are more difficult for children to understand and may require explicit instruction. In relation to the operations of addition and subtraction, Nunes et al. (2009) proposed that explicit teaching of concepts, such as inversion and their associated conceptually-based shortcuts is important. Although teaching children about concepts, such as inversion or associativity in the classroom might present a challenge to researchers who rely on the novelty of the problems to assess use of the conceptually-based shortcuts, the benefits to children clearly far outweigh the costs to researchers. By explicitly drawing children’s attention to how operations are related to one another and how conceptual knowledge can be applied via problem solving procedures are likely important steps toward increasing children’s knowledge of mathematics (Schneider & Stern, 2009). It can also help reinforce the need to look for patterns and relations—a central aspect of mathematics (Lai et al., 2008). For children to gain a deep understanding of mathematics, children may need to be taught not only about facts and procedures, but also about concepts as outlined...
The studies discussed in this chapter not only highlight the importance of arithmetic concepts but also reveal some important research and educational issues. From a research perspective, it should be clear that more attention should be paid to multiplicative concepts. They are more challenging for children than additive concepts and take much longer to develop. Further, to gain a more complete understanding of how concepts develop, whether they are additive or multiplicative, different patterns of development may be identified depending on the concept being investigated. This expanded research agenda should also include a focus on developmental differences in the emergence of additive and multiplicative concepts. Finally, given the striking individual variability from the early elementary years to the university years in how well arithmetic concepts are understood and applied during problem solving, a more thorough examination is needed of the factors or, better yet, the combination of factors relating to the understanding of arithmetic concepts and the application of conceptually-based shortcuts.

From an educational perspective, teachers and parents need to be more aware that what may seem obvious to them (e.g., that multiplication and division are the inverse operations of each other), is not necessarily obvious to children. Just as teachers overestimate their students' understanding of the equal sign (Sherman, 2007), teachers may overestimate their students' understanding of simple concepts including inversion and associativity. By teaching children about the relations between operations and by directly comparing and contrasting additive and multiplicative concepts, children will have the opportunity to become deeper and more flexible mathematical thinkers—qualities that will be invaluable as they move from dealing with simple arithmetic problems and concepts to more complex mathematical concepts and ideas.

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**INTRODUCTION**

Solving a word problem requires the correct use of arithmetic concepts and the application of the right procedure, as described in the text. Problem solving is described as a mathematization process, which involves translation of a word problem solving task into a mathematical problem. The comprehension of a word problem solving task involves the selection of the appropriate mathematical framework, which is often influenced by theories of arithmetic understanding. These theories share the assumption that young children have limited understanding of fundamental mathematical concepts, or to right for math problems, but, unfortunately, these frameworks are sometimes disregarded concrete instruction.