Bailey, D. H., Siegler, R. S., & Geary, D. C. (2014). Early predictors of middle school fraction knowledge. *Developmental Science, 17,* 775-785. doi: 10.1111/desc.12155

SHORT REPORT

Early predictors of middle school fraction knowledge

Drew H. Bailey[1], Robert S. Siegler[1,2], and David C. Geary[3]

[1] *Carnegie Mellon University*
[2] *Siegler Center for Innovative Learning, Beijing Normal University*
[3] *University of Missouri*

Abstract

Recent findings that earlier fraction knowledge predicts later mathematics achievement raise the question of what predicts later fraction knowledge. Analyses of longitudinal data indicated that whole number magnitude knowledge in first grade predicted knowledge of fraction magnitudes in middle school, controlling for whole number arithmetic proficiency, domain general cognitive abilities, parental income and education, race, and gender. Similarly, knowledge of whole number arithmetic in first grade predicted knowledge of fraction arithmetic in middle school, controlling for whole number magnitude knowledge in first grade and the other control variables. In contrast, neither type of early whole number knowledge uniquely predicted middle school reading achievement. We discuss the implications of these findings for theories of numerical development and for improving mathematics learning.

Introduction

Fractions and the closely related concepts of decimals, percentages, ratios, rates, and proportions are omnipresent in algebra, geometry, statistics, chemistry, physics, and other areas of mathematics and science. These subjects literally cannot be understood without understanding rational numbers. By precluding mastery of key areas of mathematics and science, poor understanding of fractions and other expressions of rational numbers also precludes later participation in many stimulating and remunerative occupations (McCloskey, 2007).

Learning fractions also enables children to deepen their understanding of numbers. In particular, it allows children to distinguish between properties shared by all real numbers and properties that are invariant for whole numbers but not for real numbers in general. All whole numbers have unique predecessors and successors, increase with multiplication, decrease with division, and can be represented by a single symbol. None of these properties, however, holds true for fractions. Instead, the one property that unites fractions, whole numbers, and indeed all real numbers is that they represent magnitudes that can be ordered on a number line. This analysis underlies Siegler, Thompson, and Schneider's (2011) integrated theory of numerical development, which proposes that fractions play a key role in learning the properties that are and are not invariant for real numbers, and thus that learning fractions plays a crucial role in acquiring a mature concept of number.

Address for correspondence: Drew H. Bailey, Psychology Department, Carnegie Mellon University, 5000 Forbes Ave., Pittsburgh, PA 15213, USA; email: drewhalbailey@cmu.edu or David C. Geary, Department of Psychological Sciences, University of Missouri, 210 McAlester Hall, Columbia, MO 65211; email: GearyD@missouri.edu

Consistent with the view that understanding fractions is central to subsequent mathematics learning, early knowledge of fractions is highly predictive of much later mathematics achievement. Longitudinal data from both the U.S. and the U.K. indicate that knowledge of fractions at age 10 predicts knowledge of algebra and overall mathematics achievement at age 16, even after statistically controlling for IQ, reading achievement, working memory, family income and education, and whole number arithmetic (Siegler, et al., 2012). In both countries, concurrent correlations between 16-year-olds' fraction knowledge and their overall mathematics achievement exceeded $r = .80$. Shorter-term longitudinal studies have shown similar relations (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012).

The key role of fractions in mathematics makes it especially unfortunate that many children have little understanding of them (Vamvakoussi & Vosniadou, 2004, 2010). To cite one example of the problem, 50% of a nationally representative sample of U.S. eighth graders failed to correctly order from smallest to largest the fractions 2/7, 5/9, and 1/12 (Martin, Strutchens, & Elliott, 2007). Consistent with these standardized test data, a sample of 1,000 U.S. algebra teachers rated weak fraction knowledge the second worst problem (following word problems) among 15 proposed deficiencies in their students' preparation for learning algebra (National Mathematics Advisory Panel, 2008). The difficulty extends well beyond the U.S.; children in countries with far higher mathematics achievement, such as Japan and Taiwan, also have difficulty learning fractions (Chan, Leu, & Chen, 2007; Yoshida & Sawano, 2002). Moreover, poor understanding of fractions often persists into adulthood; a sample of U.S. community college students correctly answered only 70% of two-choice fraction magnitude comparison problems, where chance was 50% correct (Schneider & Siegler, 2010).

One reason why learning fractions is difficult is the whole number bias, the interfering effect of whole number knowledge (Gelman & Williams, 1998; Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2004, 2010). With regard to magnitude understanding, this negative influence is evident when children base fraction comparisons on the whole number expressed in the numerator, for example by reasoning that 5/9 > 2/3 because 5>2 (Meert, Gregoire, & Noel, 2009; 2010). With regard to arithmetic procedures, the negative influence is evident in the frequency of errors such as 2/3 + 3/4 = 5/7, in which numerators and denominators of fractions are treated as independent whole numbers (Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2004). As these examples indicate, whole number knowledge can, and often does, interfere with fraction performance at a given point in time. In the long term, however, superior whole number knowledge might positively influence *learning* of fractions, a possibility that we tested here.

In the present study, we examined whether it is possible to predict early in formal schooling which children will have difficulty learning fractions and to identify specific developmental antecedents of fraction difficulties. In particular, we attempted to identify aspects of first graders' whole number knowledge that predict specific aspects of their fraction knowledge in middle school. Previous longitudinal studies (Duncan, et al., 2007; Stevenson & Newman, 1986) have established that overall mathematics achievement early in elementary school is predictive of much later overall mathematics achievement. The present study was an attempt to build on these findings to determine which types of early mathematical understandings are predictive of one central part of later mathematics understanding, fractions.

With both whole numbers and fractions, we separately examined two main components of mathematical understanding: conceptual knowledge and procedural knowledge. Conceptual understanding of numbers consists of semantic knowledge of the properties of the numbers, including the magnitudes they represent, the principles that underlie their use, and the notation in which they are expressed. Once acquired, conceptual knowledge sometimes is generalized quite broadly and can be useful on novel tasks (Siegler & Crowley, 1994). In contrast, procedural knowledge of numbers involves skilled execution of specific procedures for solving the four arithmetic operations. It usually is task-specific, rarely generalizing to novel tasks (Anderson & Lebiere, 1998). The difference between conceptual and procedural knowledge of numbers can be illustrated in the context of fraction division. Many adults possess the procedural knowledge needed to efficiently execute the invert and multiply algorithm for fraction division, but only a small subset have the conceptual knowledge needed to explain why the procedure works or to estimate the magnitude of the answer (Ball, 1990; Ma, 1999).

The distinction between conceptual and procedural knowledge of numbers is an important one in the context of the current study because both conceptual and procedural knowledge are essential to understanding both whole numbers and fractions (Hiebert & LeFevre, 1986; Rittle-Johnson & Siegler, 1998), because they are independently predictive of mathematics achievement (Bailey, et al., 2012a; Jordan, et al., 2013; Rittle-Johnson & Alibabli, 1999), and because the two types of numerical knowledge seem likely to develop in different ways. Acquisition of procedural knowledge generally arises through repeated practice of a fixed sequence of steps. Such practice slowly speeds the execution of the procedure and reduces the cognitive resources required for its execution (Anderson & Lebiere, 1998; Bailey, Littlefield, & Geary, 2012; Fuchs et al., 2013a). In contrast, acquisition of conceptual knowledge involves improved semantic encoding, in which new or existing pieces of information are related to existing knowledge within a semantic network (Anderson & Lebiere, 1998). Acquisition of conceptual knowledge does not necessarily require much practice. A single event, such as hearing a compelling explanation or having an insight, can produce a large increase in conceptual knowledge (Cooper & Sweller, 1987; Sternberg & Davidson, 1995).

We assessed conceptual knowledge of whole numbers and fractions through measures of symbolic magnitude understanding, that is, measures in which whole numbers and fractions are presented as numerals rather than as dot collections. Measures of symbolic magnitude understanding, rather than of non-symbolic magnitude understanding, were used because symbolic magnitude understanding is more strongly related to mathematics achievement (see review by De Smedt, Noel, Gilmore, & Ansari, 2013). Henceforth, when we refer to magnitude knowledge and tasks measuring that knowledge, we are referring specifically to their knowledge of symbolic magnitudes.

Conceptual knowledge was assessed by presenting symbolic magnitude comparison and number line estimation tasks. The magnitude comparison task required children to choose the larger of two numbers; the number line estimation task required children to estimate the location of a number relative to numbers at the ends of a line. Consistent with the hypothesis that these tasks measure the same underlying construct, they have been shown to correlate highly for both whole numbers (Laski & Siegler, 2007; Ramani & Siegler, 2008) and fractions (Siegler & Pyke, 2013; Siegler et al., 2011).

Procedural knowledge of both whole numbers and fractions was measured by addition performance, which has been found to be correlated with skill at other arithmetic operations (Siegler & Pyke, 2013) as well as being predictive of overall mathematics achievement (Geary, 2011; Geary et al., 2009).

The main hypotheses of the present study were that after statistically controlling for IQ, executive functioning, parental income, parental education, race, and gender: 1) Early (first grade) knowledge of whole number magnitudes will predict much later (seventh and eighth grade) knowledge of fraction magnitudes; 2) Early knowledge of whole number arithmetic will predict much later knowledge of fraction arithmetic; 3) Early knowledge of whole number magnitudes will predict much later knowledge of fraction arithmetic, with the relation mediated by knowledge of fraction magnitudes; 4) Early knowledge of whole number arithmetic will not predict much later knowledge of fraction magnitudes; and 5) Early whole number magnitude and arithmetic knowledge will not predict a later non-mathematical academic outcome, reading achievement.

There were several reasons to expect that early knowledge of whole number magnitudes would predict later knowledge of fraction magnitudes. Acquisition of a mental number line that includes fractions is akin to filling in the empty spaces between successive whole numbers. Children who form precise representations of whole number magnitudes on the mental number line seem likely to also form precise representations of fraction magnitudes on it. One reason is that precise magnitude representations of whole numbers and fractions require the same type of encoding of each number relative to other numbers; for example, just as 75 is 75% of the way from 0-100, so 3/4 is 75% of the way from 0-1. Children may not always encode the magnitude of numbers. For example, 2/7 could be encoded in ways irrelevant to magnitude – as containing the digit 2, the digit 7, or the digits 2 and 7 – or it could be encoded in ways relevant to magnitude – as less than 1/2, close to .3, or roughly halfway between 0 and 1/2.

Another potential reason to predict positive relations between early individual differences in whole number magnitude knowledge and much later differences in fraction magnitude knowledge is that fraction and whole number magnitude representations can be linked through the intermediary of decimals. Accurately translating fractions to decimals would make comparing them or locating them on number lines very similar to doing the same with the corresponding whole number problems.

Our second hypothesis was that proficiency at early whole number arithmetic would predict later proficiency at fraction arithmetic. The main reason is that children who gain early proficiency in whole number arithmetic automatize execution of those arithmetic procedures, which would free working memory resources for learning and correctly executing fraction arithmetic procedures (Geary, 2011). Another reason is that about 10% of children's errors on fraction arithmetic problems are caused by whole number arithmetic errors (Siegler & Pyke, 2013; Siegler, et al., 2011), and children who are skilled at whole number arithmetic would make fewer such errors.

The third hypothesis was that early whole number magnitude knowledge would positively predict much later fraction arithmetic knowledge, with the relation mediated by fraction magnitude knowledge. That is, we hypothesized that early whole number magnitude knowledge enhances acquisition of later fraction magnitude knowledge, which in turn enhances acquisition of later fraction arithmetic knowledge. In addition to the hypothesized positive relation between early whole number magnitude knowledge and later fraction magnitude knowledge, such a mediational relation would require a positive relation between knowledge of fraction magnitudes and fraction arithmetic. Several studies have documented that these types of knowledge are positively correlated (Hecht, et al., 2003; Hecht & Vagi, 2010; Siegler, et al., 2011). Moreover, Fuchs et al. (2013b) found in an intervention study that the experimental group’s gains in fraction arithmetic knowledge were mediated by increases in their fraction magnitude knowledge.

These three hypotheses all reflect the assumption of the integrated theory of numerical development that understanding of fractions is continuous with understanding of whole numbers, and that individual differences in the two should be related, independent of domain general abilities such as IQ and executive functioning (Sielger et al.,2011). This is not necessarily the case, however. Gaining a conceptual understanding of fractions is far more demanding than gaining a similar understanding of whole numbers. Reflecting this difference, sixth and eighth graders’ number line estimation with fractions is less accurate than *first graders’* number line estimation with whole numbers (Booth & Siegler, 2008; Siegler & Booth, 2004; Siegler & Pyke, 2013; Siegler, et al., 2011). The processing demands of combining numerator and denominator into a single integrated magnitude, of realizing that fraction magnitudes decrease rather than increase with increasing denominator size, and of arithmetically combining unlike units (e.g., adding thirds and sevenths, as in 2/3 + 3/7) might lead to domain general cognitive abilities, rather than whole number magnitude knowledge, being the main determinant of individual differences in fraction magnitude knowledge. Similarly, it was by no means a foregone conclusion that children who are skilled at whole number arithmetic would also be skilled at fraction arithmetic. While the large majority of whole number arithmetic errors are fact retrieval errors, the large majority of children's errors on fraction arithmetic problems arise from the use of inappropriate problem solving procedures, including using whole number arithmetic strategies independently on the numerator and the denominator (e.g., adding the numerators and denominators in a fraction addition problem; Siegler et al., 2011; Siegler & Pyke, 2013). Learning facts and learning procedures, especially procedures whose conceptual basis is hard to understand for many learners, involve quite different cognitive processes, and thus may not be robustly related, controlling for domain general cognitive abilities.

A fourth hypothesis was that early knowledge of whole number arithmetic would be unrelated to later fraction magnitude knowledge. The reason was that no obvious path led from mastery of the one to mastery of the other.

The fifth hypothesis was that whole number magnitude and arithmetic knowledge would not uniquely predict reading achievement. The purpose of examining this relation was to test whether early whole number magnitude and arithmetic knowledge predict all later academic outcomes, as might occur if they were either supplementary measures of general intellectual ability or indices of academic motivation. Prior findings relevant to this issue have been mixed. Duncan et al. (2007) found that kindergartners' mathematics knowledge uniquely predicted fifth graders' reading, above and beyond the kindergartners' domain general cognitive abilities, early reading, and family income and education. However, Siegler et al. (2012) found that fifth graders' mathematics knowledge did not predict tenth graders' reading, when controlling for the same categories of variables. The integrated theory of numerical development provided no obvious reason to expect that early whole number magnitude and arithmetic knowledge would predict later reading beyond the effects of the control variables. Therefore, we predicted that first graders' whole number knowledge would be unrelated, or at most weakly related, to their reading outcomes in middle school, after controlling for demographic and domain general cognitive variables.

Method

Participants

The participants were 162-172 children from a longitudinal study of mathematical development (Geary, 2010; Geary, et al., 2007) who performed the relevant tasks in first, seventh, and eighth grades (the exact number of children varied with the analysis). See Appendix A of the Supporting Information for additional information on the sample.

Tasks

This section provides a brief description of each task that we examined. Details of the tasks, procedures, and measures are provided in the Supporting Information, Appendix B. The reason that seventh grade data are reported on some measures and eighth grade data on others is that the tasks were presented either in seventh or eighth grade but not both (the one exception was the reading achievement test, where an arbitrary decision was made to use the eighth rather than the seventh grade data). In cases where two tasks assessed the same construct in the same grade, we generated composite measures from the two, as indicated below.

Reading achievement

Eighth graders' reading achievement was assessed using number of correct answers on the Word Reading subtest of the *Wechsler Individual Achievement Test-II-Abbreviated* (Wechsler, 2001). Means and standard deviations of the outcome measures appear in the Supporting Information, Table S1.

Intelligence

IQ in first grade was assessed with the *Wechsler Abbreviated Scale of Intelligence* (Wechsler, 1999).

Working memory

Central executive functioning in first grade was assessed by number of correct answers on the Listening Recall, Counting Recall, and Backward Digit Recall subtests from the *Working Memory Test Battery for Children* (WMTB-C; Pickering & Gathercole, 2001).

Spatial abilities

Children also completed the Block Recall and Mazes Memory subtests, measures of the visuospatial sketchpad, from the WMTB-C. The number of correct answers on these tasks were summed and used as a measure of spatial abilities.

Whole number arithmetic

Two tests were used to create a composite measure of first graders' arithmetic knowledge. One was the retrieval test, which involved 14 problems with single-digit addends; the number of problems on which children answered correctly and said that they retrieved the answer was used to measure whole number arithmetic proficiency. Retrieval is the fastest addition strategy, and the ability to correctly retrieve simple addition problems has been shown to predict elementary school students' mathematical achievement trajectories (Geary, 2011; Geary et al., 2009).

The second measure of early arithmetic knowledge was the Numerical Operations subtest of the *Wechsler Individual Achievement Test-II-Abbreviated* (Wechsler, 2001). Number of correct answers was the measure of competence on this test. The composite whole number arithmetic score was created by standardizing scores on the two tasks separately and then calculating each child's mean standardized score.

Fraction arithmetic

Seventh graders were provided 1 minute to solve 12 fraction addition problems and 1 minute to solve 12 fraction division problems. Total number of problems solved correctly was the measure of fraction arithmetic proficiency.

Whole number magnitudes

First graders completed a 24-item number line estimation task with whole numbers from 0-100. Estimation accuracy was assessed using percent absolute error – the percentage deviation of each estimate from the correct placement of that number on the number line – multiplied by -1, because on this measure, higher scores indicated less accurate performance. The transformation allowed positive correlations to consistently indicate that more accurate performance on different tasks went together.

Fraction magnitudes

Eighth graders' fraction magnitude knowledge was assessed through performance on two tasks. The fraction magnitude comparison task involved circling the larger of two fractions on 48 items within 90 s. Performance was measured as number of correct answers minus number of incorrect answers.

Fraction number line estimation involved approximating the location of 10 fractions on a 0-5 number line. Performance on each task was assessed in the same way as on the corresponding task with whole numbers, and the composite measure of magnitude knowledge was also generated in the same way.

Results

Relations between early whole number knowledge and later fraction knowledge

We computed bivariate correlations and linear regression analyses to examine whether first graders' knowledge of whole number magnitudes and arithmetic predicted their knowledge of fraction magnitudes, fraction arithmetic, and reading achievement six or seven years later. The regression weights were estimated controlling for the following variables assessed in first grade: IQ, executive function, parental income, parental education, race, and gender, as well as for the type of whole number knowledge that was not the dependent variable in the analysis. Bivariate correlations are reported in Table 1; linear regression analyses are reported in Table 2. To lessen the probability of Type I error, we set an alpha of .01.

Table 1: Correlation Matrix

	Frac Mag 8	Frac Arith 7	Read Ach 8	IQ 1	Cent Exec 1	WN Mag 1	WN Arith 1	Parent Ed	Income	Race (A.A.)	Race (Other)
Frac Mag 8											
Frac Arith 7	0.62										
Read Ach 8	0.54	0.36									
IQ 1	0.49	0.34	0.54								
Cent Exec 1	0.50	0.29	0.53	0.42							
WN Mag 1	0.62	0.45	0.37	0.44	0.49						
WN Arith 1	0.56	0.48	0.38	0.37	0.42	0.55					
Parent Ed	0.30	0.23	0.45	0.26	0.25	0.23	0.17*a*				
Income	0.35	0.28	0.36	0.23	0.12	0.19	0.12	0.57			
Race (A.A.)	-0.22	-0.19	-0.18	-0.29	-0.27	0.24	-0.17*a*	-0.01	-0.07		
Race (Other)	-0.24	0.02	-0.13	-0.04	-0.13	0.11	0.00	-0.11	-0.01	-0.19*a*	
Gender (Male)	0.26	0.22	0.10	0.13	-0.03	0.22	0.22	0.11	0.07	0.05	-0.05

Note: Frac = Fraction; WN = Whole Number; Mag = Magnitude Knowledge; Arith = Arithmetic Knowledge; Read Ach = Reading Achievement; Cent Exec = Central Executive; Parent Ed. = Parental Education; A. A. = African American. Numbers indicate the grade in which a variable was assessed. In all cases in which $|r| > .19$ or the correlation is followed by *"a"*, $p < .01$.

As hypothesized, after controlling for the other first grade predictors, first graders' whole number magnitude knowledge predicted eighth graders' fraction magnitude knowledge, $\beta = .27$, $p < .001$ (Table 2, left column). First graders' whole number arithmetic knowledge, $\beta = .25$, $p < .001$, and central executive functioning, $\beta = .22$, $p < .001$, also predicted eighth graders' fraction magnitude knowledge.

Because number line estimation tasks involve mapping numerical magnitudes to a physical space, we tested whether individual differences in spatial abilities could account for the predictive relation between first graders' number line estimation accuracy with whole numbers and their fraction magnitude knowledge in eighth grade. To do so, we added first graders' spatial abilities as a predictor within the previously described regression model. First graders' spatial ability did not predict their fraction magnitude knowledge in middle school, nor did controlling for spatial ability affect the relation between first graders' whole number magnitude knowledge and their fraction magnitude knowledge in middle school (Supporting Information, Table S2).

Another potential explanation of the continuity between early whole number and later fraction magnitude knowledge was the use of number line tasks at both times of measurement. To test this interpretation, we divided the composite dependent measure of middle school fraction magnitude knowledge into its two components: fraction magnitude comparison accuracy and fraction number line estimation accuracy – and ran parallel regression analyses with each dependent measure. As shown in the Supporting Information, Table S3, first graders' number line estimation accuracy with whole numbers was just as predictive of later fraction magnitude comparison accuracy as of later fraction number line estimation accuracy. Thus, use of the number line task as a measure of early whole number magnitude knowledge and as one of two measures of later fraction magnitude knowledge did not explain the relation between the two.

Table 2: Early Predictors of Middle School Fractions and Reading

Outcome:	Eighth Grade Fraction Magnitude Knowledge	Seventh Grade Fraction Arithmetic Knowledge	Eighth Grade Reading Achievement
First Grade Predictor			
Domain General Cognitive Abilities:			
IQ	.11 (.07)	.09 (.08)	.29** (.07)
Central Executive	.22** (.07)	.07 (.08)	.29** (.07)
Whole Number Magnitude Knowledge:			
Number Line	.27** (.07)	.20 (.08)	.05 (.07)
Whole Number Arithmetic Knowledge:			
Arithmetic Composite	.25** (.06)	.29** (.08)	.14 (.07)
Demographic Variables:			
Parental Education	-.09 (.07)	-.03 (.08)	.24* (.07)
Household Income	.16 (.07)	.14 (.08)	.08 (.07)
Race (African Amer)	-.06 (.06)	-.04 (.07)	-.02 (.06)
Race (Other)	-.13 (.06)	.09 (.07)	-.02 (.06)
Gender (Male)	.09 (.06)	.05 (.07)	.00 (.06)
N	162	172	162
R^2	.57	.33	.50

Note: * $p < .01$, ** $p < .001$. Regression weights come from linear regression models, which include all predictor variables in the table. All regression weights are standardized, with standard errors in parentheses.

A parallel linear regression analysis showed that as expected, after controlling for the other variables measured in first grade, fraction arithmetic in middle school was uniquely predicted by whole number arithmetic in first grade, $\beta = .29$, $p < .001$ (Table 2, middle column). The hypothesized predictive relation between first graders' whole number magnitude knowledge and middle school children's fraction arithmetic fell just short of the .01 alpha level, $\beta = .20$, $p = .019$.

A third, parallel, linear regression analysis indicated that after controlling for the other variables, reading achievement in eighth grade was not uniquely predicted by either whole number variable in first grade. Instead, it was predicted by first graders' IQ, $\beta = .29$, $p < .001$, central executive functioning, $\beta = .29$, $p < .001$, and parental education, $\beta = .24$, $p < .01$ (Table 2, right column). These relations indicated that the predictive power of early whole number knowledge did not extend to middle school intellectual attainments in general.

Mediation analyses

To test whether, as hypothesized, fraction magnitude knowledge in middle school mediated the relation between whole number magnitude knowledge in first grade and fraction arithmetic knowledge in middle school, we performed two mediation analyses. Because this analysis concerned only a single predictor (first graders' whole number magnitude knowledge), we used an alpha of .05. As in the other analyses, we controlled for all of the other measures included in Table 2. The steps we took to perform the mediation analyses are described in the Supporting Information, Appendix C.

Table 3: Mediation Tests

Initial Variable	Mediator	Outcome	Path a (S.E.)	Path b (S.E.)	Total Effect (95% CI)	Indirect Effect (95% CI)	Direct Effect (95% CI)	Proportion via Mediation
WN Mag	Frac Mag	Frac Arith	.27** (.07)	.50** (.09)	.21** (.06, .35)	.13* (.04, .24)	.08 (-.08, .24)	.62
WN Mag	Frac Arith	Frac Mag	.21* (.09)	.32** (.06)	.27** (.10, .41)	.07* (.02, .12)	.20** (.03, .34)	.26

Note: * $p < .05$; ** $p < .01$. WN Mag = First Grade Whole Number Magnitude Knowledge, Frac Mag = Middle School Fraction Magnitude Knowledge, Frac Arith = Middle School Fraction Arithmetic Knowledge. Path a is the path from the initial variable to the mediator; Path b is the path from the mediator to the outcome. The 95% confidence intervals were calculated using a nonparametric bootstrap using 1000 bootstrap iterations.

As shown in the first row of Table 3, the relation between first graders' whole number magnitude knowledge and their fraction arithmetic knowledge when they were in middle school was fully mediated by their middle school fraction magnitude knowledge. That is, first graders' whole number magnitude knowledge exercised an indirect effect on middle school children's fraction arithmetic knowledge through the mediator of their middle school fraction magnitude knowledge, $\beta = .13$, $p < .05$. There was no direct relation between first graders' whole number magnitude knowledge and their later fraction arithmetic knowledge. The parameter estimates for all predictors from this model appear in the Supporting Information, Table S4.

To test whether each type of middle school fraction knowledge mediated acquisition of the other type of knowledge, we examined whether middle school fraction arithmetic knowledge mediated the relation between first graders' whole number magnitude knowledge and their fraction magnitude knowledge in middle school. In this model (Table 3, second row), first graders' whole number magnitude knowledge exercised an indirect effect on their fraction magnitude knowledge in middle school, $\beta = .07$, $p < .05$, but a larger direct effect remained, $\beta = .20$, $p < .01$. This finding indicates that not all types of numerical knowledge fully mediate development of other types of numerical knowledge.

Finally, to rule out the possibility that the strong relations between early whole number magnitude knowledge and later fraction magnitude knowledge were due to number line tasks being used in both, we ran the mediation models with fraction magnitude comparison accuracy as the sole measure of middle school fraction magnitude knowledge (rather than including middle school number line estimation in the composite measure of fraction magnitude knowledge.) The same set of indirect and direct effects emerged as in Table 3 (Supporting Information, Table S5), indicating that the relation between early whole number magnitude knowledge and later fraction magnitude knowledge is not attributable to use of number line tasks at both times.

Discussion

The present findings were consistent with four of our five hypotheses. Consistent with the first hypothesis, first graders' knowledge of whole number magnitudes predicted their knowledge of fraction magnitudes in eighth grade, even after controlling for the first graders' whole number arithmetic knowledge, IQ, central executive functioning, parental education, household income, race, and gender. Consistent with the second hypothesis, first graders' knowledge of whole number arithmetic predicted their knowledge of fraction arithmetic in seventh grade, controlling for the same variables and whole number magnitude knowledge. Consistent with the third hypothesis, middle school fraction magnitude knowledge mediated the relation between first grade whole number magnitude knowledge and middle school fraction arithmetic. Consistent with the fifth hypothesis, these findings were not due to early whole number magnitude and arithmetic knowledge predicting middle school academic outcomes in general; neither type of whole number knowledge measured in first grade uniquely predicted middle school reading achievement.

The one hypothesis that was disconfirmed by the present findings concerned a relation that was not expected to be present but that was: First graders' knowledge of whole number arithmetic predicted their knowledge of fraction magnitudes in eighth grade. Although conceptual and procedural knowledge of mathematics often exert a bidirectional influence (Rittle-Johnson & Siegler, 1998), the means through which whole number arithmetic knowledge affect fraction magnitude knowledge remain unclear. An unmeasured variable general to mathematics learning, such as motivation to learn mathematics, attentiveness to classroom mathematics instruction, or overall mathematics ability, might account for the relation, but this remains to be determined.

The relation between first graders' knowledge of whole number magnitudes and their knowledge of fraction arithmetic in middle school was fully mediated by their knowledge of fraction magnitudes in middle school. This is consistent with the hypothesis, which followed from the integrated theory of numerical development (Siegler et al., 2011), that early whole number magnitude knowledge facilitates later acquisition of fraction magnitude knowledge, which in turn facilitates later acquisition of fraction arithmetic knowledge. Thus, results from both the present longitudinal study and a previous intervention study (Fuchs, et al., 2013b) converge on the conclusion that fraction magnitude knowledge promotes fraction arithmetic skill.

The mediation analyses also demonstrated the value of distinguishing between conceptual and procedural knowledge of mathematics. The distinction allowed us to go beyond the finding that early knowledge of whole numbers predicts later knowledge of fractions to obtain the more nuanced finding that the relation between early whole number conceptual knowledge and later fraction procedural knowledge is mediated by fraction conceptual knowledge. Assessing other types of conceptual knowledge, such as knowledge of mathematical principles, might further enrich the developmental account.

First graders' central executive functioning uniquely predicted middle school children's fraction magnitude knowledge but not their fraction arithmetic knowledge. One likely reason was differences in amount of previous experience with the tasks. Domain general cognitive abilities have often been found to be more strongly related to performance on novel tasks than to performance on highly practiced ones (Ackerman, 1988, 2007, 1992; Geary, 2005). In the present context, children almost certainly receive far less practice on fraction magnitude comparison and number line estimation than on fraction arithmetic. These differences in amount of practice seem likely to contribute to first graders' central executive functioning being more closely related to their later performance on the relatively unfamiliar fraction magnitude tasks than to their performance on more familiar fraction arithmetic tasks.

Several mechanisms might plausibly account for the long-term consistencies in individual differences in magnitude and arithmetic knowledge that were found. Commonalities in the amount and precision of encoding the magnitudes of fractions and whole numbers in relation to other numbers might have linked individual differences in early whole number magnitude knowledge to much later individual differences in fraction magnitude knowledge. Translation of fractions into decimals, and using the decimals like whole numbers on the fraction magnitude comparison and number line estimation tasks, provided another mechanism that could link the two types of magnitude representations.

The long-term continuity between whole number and fraction arithmetic knowledge seems likely to involve some of the same mechanisms as in the developmental continuity in magnitude knowledge, and some different ones. The mechanisms that directly support learning of both whole number and fraction magnitudes seem likely to indirectly support learning of both whole number and fraction arithmetic. Knowledge of magnitudes and arithmetic is correlated for both whole numbers and fractions, so the mechanisms that produce correlations between knowledge of fraction and whole number magnitudes would indirectly tend to produce correlations between whole number and fraction arithmetic. Another mechanism that seems likely to produce relations between early whole number arithmetic and later fraction arithmetic is correlations in amount of practice at each skill. Amounts of practice in whole number and fraction arithmetic could be correlated due to children's school emphasizing (or not emphasizing) arithmetic practice, children's conscientiousness (or lack of such) in doing their whole number and fraction arithmetic homework, and parents' engagement (or lack of such) in insuring that their children do both types of homework. Also, if individual differences in whole number arithmetic are stable over time, poor whole number arithmetic skill will directly cause some errors in fraction arithmetic, again leading to relations between early whole number arithmetic and later fraction arithmetic.

The present findings also have implications for mathematics education. U.S. mathematics curricula have often been critiqued for being "a mile wide and an inch deep" (e.g., Schmidt, Houang, & Cogan, 2002). Partially in response to this critique, the central recommendation of the National Mathematics Advisory Panel (2008) was that mathematics instruction in elementary and middle school focus on promoting knowledge of whole numbers and fractions. The present longitudinal data, together with the longitudinal data of Siegler, et al. (2012), lend support to this instructional recommendation. The two studies indicate that first graders' knowledge of whole number magnitudes and arithmetic are uniquely predictive of seventh and eighth graders' knowledge of fraction magnitudes and arithmetic, and that fifth graders' knowledge of fractions and whole number division are uniquely predictive of tenth graders' overall mathematics achievement and algebra knowledge. These findings are consistent with the trend, embodied by the Common Core State Standards Initiative (2010) toward focusing on fewer mathematical goals and pursuing them in more depth.

The studies also point to a way in which longitudinal studies can further inform educational policy. In addition to strands involving whole numbers and rational numbers, the Common Core State Standards also has strands involving geometry and data analysis/measurement. Implementation of all four strands begins in kindergarten and continues in each grade thereafter. At present, no data are available regarding whether early proficiency in geometry and data analysis/statistics uniquely contributes to later mathematics achievement, much less regarding the areas of mathematics to which they contribute. Analyses of longitudinal data sets can help answer these questions and thus inform decisions regarding which areas to emphasize in elementary and middle school mathematics education.

Acknowledgements

This research was supported by the Eunice Kennedy Shriver National Institute of Child Health and Human Development Grants R01 HD38283 and R37 HD045914, by the Department of

Education Grant R324C10004, as well as by the Teresa Heinz Chair at Carnegie Mellon University. We thank Mary Hoard, Lara Nugent, Linda Coutts, Chip Sharp, Jennifer Byrd-Craven, Chatty Numtee, Amanda Shocklee, Sara Ensenberger, Kendra Andersen Cerveny, Rebecca Hale, Patrick Maloney, Ashley Stickney, Nick Geary, Mary Lemp, Cy Nadler, Mike Coutts, Katherine Waller, Rehab Mojid, Jasmine Tilghman, Caitlin Cole, Leah Thomas, Erin Twellman, Patricia Hoard, Jonathan Thacker, Alex Wilkerson, Stacey Jones, James Dent, Erin Willoughby, Kelly Regan, Kristy Kuntz, Rachel Christensen, Jenni Hoffman, and Stephen Cobb for help on various aspects of the project. We also thank Lynn Fuchs and Nancy Jordan for comments on an earlier draft of this manuscript.

References

Ackerman, P. L. (1988). Determinants of individual differences during skill acquisition: Cognitive abilities and information processing. *Journal of Experimental Psychology: General*, *117*, 288-318.Ackerman, P. L. (1992). Predicting individual differences in complex skill acquisition: dynamics of ability determinants. *Journal of Applied Psychology*, *77*, 598-614.

Ackerman, P. L. (2007). New developments in understanding skilled performance. *Current Directions in Psychological Science*, *16*, 235-239.Anderson, J. R. & Lebiere, C. (1998). *The atomic components of thought.* Mahwah, NJ: Erlbaum.

Anderson, J. R. & Lebiere, C. (1998). *The atomic components of thought.* Mahwah, NJ: Erlbaum.

Bailey, D. H., Hoard, M. K., Nugent, L., & Geary, D. C. (2012a). Competence with fractions predicts gains in mathematics achievement. *Journal of Experimental Child Psychology, 113*, 447-455.

Bailey, D. H., Littlefield, A., & Geary, D. C. (2012b). The co-development of skill at and preference for use of retrieval-based processes for solving addition problems: Individual and sex differences from first to sixth grade. *Journal of Experimental Child Psychology, 113,* 78-92.

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal of Research in Mathematics Education*, *21*, 132-144.

Booth, J. L., & Newton, K. J. (2012). Fractions: Could they really be the gatekeeper's doorman? *Contemporary Educational Psychology, 37*, 247-253.

Booth, J. L., & Siegler, R. S. (2006). Developmental and individual differences in pure numerical estimation. *Developmental Psychology*, *41*, 189-201.

Booth, J. L., & Siegler, R. S. (2008). Numerical magnitude representations influence arithmetic learning. *Child Development, 79*, 1016-1031.

Chan, W.-H., Leu, Y.-C., & Chen, C.-M. (2007). Exploring group-wise conceptual deficiencies of fractions for fifth and sixth graders in Taiwan. *The Journal of Experimental Education, 76*, 26-57.

Common Core State Standards Initiative. (2010). *Common Core Standards for Mathematics*. Retrieved from http://www.corestandards.org/.

Cooper, G., & Sweller, J. (1987). Effects of schema acquisition and rule automation on mathematical problem-solving transfer. *Journal of Educational Psychology, 79*, 347-362.

De Smedt, B., Noel, M.P., Gilmore, C. & Ansari, D. (2013). How do symbolic and non-symbolic numerical magnitude processing relate to individual differences in children's mathematical skills? A review of evidence from brain and behaviour. *Trends in Neuroscience and Education, 2,* 48-55.

Duncan, G. J., Dowsett, C. J., Claessens, A., Magnuson, K., Huston, A. C., Klebanov, P., et al. (2007). School readiness and later achievement. *Developmental Psychology, 43*, 1428-1446.

Fuchs, L. S., Geary, D. C., Compton, D. L., Fuchs, D., Schatschneider, C., Hamlett, C. L., et al. (2013a). Effects of first-grade number knowledge tutoring with contrasting forms of practice. *Journal of Educational Psychology*, *105*, 58-77.

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Hamlett, C. L., Cirino, P. T., Jordan, N. C., Siegler, R., Gersten, R., & Changas, P. (2013b). Improving at-risk learners' understanding of fractions. *Journal of Educational Psychology, 105*, 683-700. doi: 10.1037/a0032446.

Geary, D. C. (2005). *The origin of mind: Evolution of brain, cognition, and general intelligence*. Washington, DC: American Psychological Association.

Geary, D. C. (2010). Missouri longitudinal study of mathematical development and disability. *British Journal of Educational Psychology Monograph Series II, 7*, 31-49.

Geary, D. C. (2011). Cognitive predictors of achievement growth in mathematics: A 5-year longitudinal study. *Developmental Psychology, 47*, 1539-1552.

Geary, D. C., Bailey, D. H., Littlefield, A., Wood, P., Hoard, M. K., & Nugent, L. (2009). First-grade predictors of mathematical learning disability: A latent class trajectory analysis. *Cognitive Development, 24,* 411-429.

Geary, D. C., Hoard, M. K., Byrd-Craven, J., Nugent, L., & Numtee, C. (2007). Cognitive mechanisms underlying achievement deficits in children with mathematical learning disability. *Child Development, 78*, 1343-1359.

Gelman, R., & Williams, E. (1998). Enabling constraints for cognitive development and learning: Domain specificity and epigenesis. In W. Damon, D. Kuhn & R. S. Siegler (Eds.), *Handbook of child psychology*. New York: Wiley.

Hecht, S. A., Close, L., & Santisi, M. (2003). Sources of individual differences in fraction skills. *Journal of Experimental Child Psychology, 86*, 277-302.

Hecht, S. A., & Vagi, K. J. (2010). Sources of Group and Individual Differences in Emerging Fraction Skills. *Journal of Educational Psychology, 102*, 843-859.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Erlbaum.

Jordan, C. L., Hansen, N., Fuchs, L., Siegler, R., Micklos, D., & Gersten, R. (2013). Developmental predictors of conceptual and procedural knowledge of fractions. *Journal of Experimental Child Psychology, 116*, 45-58.

Laski, E. V., & Siegler, R. S. (2007). Is 27 a big number? Correlational and causal connections among numerical categorization, number line estimation, and numerical magnitude comparison. *Child Development, 76*, 1723-1743.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers understanding of fundamental mathematics in China and the United States*. Mahwah, N.J.: Lawrence Erlbaum Associates.

Martin, W.G., Strutchens, M., & Elliott, P. (eds.) (2007). *The Learning of Mathematics, 69th NCTM Yearbook*. Reston, VA: National Council of Teachers of Mathematics.

McCloskey, M. (2007). Quantitative literacy and developmental dyscalculias. In D. B. Berch & M. M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities* (pp. 415-429). Baltimore, MD: Paul H. Brookes Publishing.

Meert, G., Gregoire, J., & Noel, M.-P. (2009). Rational numbers: Componential versus holistic representation of fractions in a magnitude comparison task. *The Quarterly Journal of Experimental Psychology, 62*, 1598-1616.

Meert, G., Gregoire, J., & Noel, M.-P. (2010). Comparing the magnitude of two fractions with common components: Which representations are used by 10- and 12-year-olds? *Journal of Experimental Child Psychology, 107*, 244-259.

National Mathematics Advisory Panel (2008) *Report of the Subcommittee on the National Survey of Algebra I Teachers*, Washington, D.C.: U.S. Department of Education.

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist, 40*, 27-52.

Pickering, S., & Gathercole, S. (2001). Working Memory Test Battery for Children (WMTB-C) manual. London: Psychological Corporation Ltd.

Ramani, G. B., & Siegler, R. S. (2008). Promoting broad and stable improvements in low-income children's numerical knowledge through playing number board games. *Child Development, 79*, 375-394.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91*, 175-189.

Rittle-Johnson, B., & Siegler, R. S. (1998). The relation between conceptual and procedural knowledge in learning mathematics: A review of the literature. In C. Donlan (Ed.), *The development of mathematical* *skills* (pp. 75-110). East Sussex, UK: Psychology Press.

Schmidt, W., Houang, R., & Cogan, L. (2002). A coherent curriculum: The case of mathematics. *American Educator*, *Summer,* 1-18.

Schneider, M., & Siegler, R. S. (2010). Representations of the magnitudes of fractions. *Journal of Experimental Psychology: Human Perception and Performance, 36*, 1227-1238.

Siegler, R. S., & Booth, J. L. (2004). Development of numerical estimation in young children. *Child Development, 75*, 428-444.

Siegler, R. S., & Crowley, K. (1994). Constraints on learning in non-privileged domains. *Cognitive Psychology, 27*, 194-227.

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., et al. (2012). Early Predictors of High School Mathematics Achievement. *Psychological Science, 23*, 691-697.

Siegler, R. S., Fazio, L. K., Bailey, D. H., & Zhou, X. (2013). Fractions: The new frontier for theories of numerical development. *Trends in Cognitive Science, 17,* 13-19.

Siegler, R. S., & Pyke, A. A. (2013, on-line). Developmental and individual differences in understanding fractions. *Developmental Psychology,* doi: 10.1037/a0031200.

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology, 62*, 273-296.

Sternberg, R. J., & Davidson, J. E. (Eds.). (1995). *The nature of insight.* Cambridge, MA: MIT Press.

Stevenson, H. W., & Newman, R. S. (1986). Long-term prediction of achievement and attitudes in mathematics and reading. *Child Development, 57*, 646-659.

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction, 14*, 453-467.

Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction, 28*, 181-209.

Wechsler, D. (1999). *Wechsler abbreviated scale of intelligence*. San Antonio, TX: PsychCorp, Harcourt Assessment, Inc.

Wechsler, D. (2001). *Wechsler individual achievement test II-abbreviated*. San Antonio, TX: The Psychological Corporation, Harcourt Brace & Co.

Yoshida, H., & Sawano, K. (2002). Overcoming cognitive obstacles in learning fractions: Equal-partitioning and equal-whole. *Japanese Psychological Research, 44*, 183-195.

Received: 25 June 2013

Accepted: 16 October 2013

Supporting Information

Additional Supporting Information may be found in the online version of this article:

Appendix A. Participants.

Appendix B. Measures and Procedures.

Appendix C. Mediation Analysis.

Table S1. Means and Standard Deviations for Outcome Measures.

Table S2. Early Predictors of Middle School Fractions and Reading, Including Spatial Abilities.

Table S3. Early Predictors of Separate Eighth Grade Fractions Magnitude Outcomes.

Table S4. Final Model from Mediation Analysis.

Table S5. Mediation Tests with Fraction Magnitude Comparison Score as Fraction Magnitude Knowledge.