Tian, J., & Siegler, R. S. (2017). Fractions learning in children with mathematics difficulties. *Journal of Learning Disabilities, 50*(6), 614-620. doi: 10.1177/0022219416662032

Fractions Learning in Children With Mathematics Difficulties

Journal of Learning Disabilities
2017, Vol 50(6), 614-620
Hammill Institute on Disabilities 2016

DOI: 10.1177/0022219416662032

Jing Tian, B.S.[1, 2], and Robert S. Siegler, Ph.D. [1, 2]

Abstract
Learning of fractions is difficult for children in general and especially difficult for children with mathematics difficulties (MD). Recent research on developmental and individual differences in fraction knowledge of MD and typically achieving (TA) children has demonstrated that U.S. children with MD start middle school behind TA peers in fraction understanding and fall further behind during middle school. In contrast, Chinese children who, like the MD children in the U.S. score in the bottom one-third of the distribution in their country, possess reasonably good fraction understanding. We interpret these findings within the framework of the integrated theory of numerical development. By emphasizing the importance of fraction magnitude knowledge for numerical understanding in general, the theory proved useful for understanding differences in fraction knowledge between MD and TA children and for understanding how knowledge can be improved. Several interventions demonstrated the possibility of improving fraction magnitude knowledge and producing benefits that generalize to fraction arithmetic learning among children with MD. The reasonably good fraction understanding of Chinese children with MD and several successful interventions with U.S. students provide hope for the improvement of fraction knowledge among American children with MD.

Keywords
mathematics difficulties, fraction learning, cross-national differences

Fractions are a critical component of mathematics understanding and a gateway to many sought-after occupations. In both the U.S. and the U.K., competence with fractions in Grade 5 uniquely predicted subsequent gains in mathematics knowledge five years later, even after statistically controlling for IQ, whole number arithmetic, and family education and income (Siegler et al., 2012). Similar predictive relations have emerged in other studies over shorter time periods and

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[1]Department of Psychology, Carnegie Mellon University, Baker Hall, 5000 Forbes Ave., Pittsburgh, PA 15213, USA
[2]Siegler Center for Innovative Learning, Beijing Normal University, Beijing 100875, China
Corresponding Author:
Jing Tian, Department of Psychology, Carnegie Mellon University, 5000 Forbes Ave., Pittsburgh, PA 15213. USA. Email: jingtianpsy2012@gmail.com

with different control variables. The importance of fractions extends beyond math classes and beyond the school years. Fractions are important for physical, biological, and social sciences and in a wide range of middle-income occupations that do not require advanced math, including nursing, carpentry, and auto mechanics (e.g., Hoyles, Noss & Pozzi, 2001; Sformo, 2008).

The importance of fractions makes it a major topic in elementary and middle school curricula. According to the Common Core State Standard Initiative (CCSSI, 2010), students should develop understanding of fraction magnitudes in Grade 3 and Grade 4, they should gain competence in fraction arithmetic and word problems from Grade 4 to Grade 6, and they should be able to apply fraction arithmetic to problems involving ratios, rates, and proportions in Grade 6 and Grade 7.

Unfortunately, even many typically achieving (TA) students do not show competence of these types after the instruction is completed. On the 2004 National Assessment of Educational Progress (NAEP; National Council of Teachers of Mathematics, 2007), 50% of 8^{th} graders failed to order three fractions (2/7, 5/9, and 1/12) from least to greatest – a skill that should be mastered in elementary school according to the CCSSI (2010). Even many adults do not understand fraction magnitudes. Among a sample of 1643 community college students who took the Mathematics Diagnostic Testing Project placement tests, only 33% correctly identified the largest of four simple fractions (Stigler, Givvin, & Thompson, 2010), barely more than the chance level of 25%.

The poor knowledge extends to understanding of fraction arithmetic. On the 1978 NAEP, when asked to choose the closest number to the sum of 7/8 + 12/13 from the list 1, 2, 19, and 21, and "don't know," the proportion of correct answers among 8^{th} graders was 24% (Carpenter, Kepner, Corbitt, Linquist, & Reys, 1980). The situation has not improved much, if at all, in the almost 40 years since 1978. Lortie-Forgues, Tian, and Siegler (2015) presented the same item and found almost identical performance, 27% correct, in a sample of 8^{th} graders from middle-income backgrounds in 2014.

As this recent finding suggests, poor fraction knowledge is not limited to standardized tests. Middle school and community college students' difficulties with fractions have been widely documented by experiments in small groups or with 1:1 experimenter-student testing procedures (Bailey et al., 2015; DeWolf, Grounds, Bassok, & Holyoak, 2014; Hecht, Close, & Santisi, 2003; Hecht & Vagi, 2010; Siegler & Pyke, 2013). To cite one example, among 6^{th} graders whose average IQ was 116, the accuracy of ranking a set of 10 fractions (including fractions with a denominator of 10 or 100) was only 59% (Mazzocco & Devlin, 2008). To cite another example, fraction arithmetic accuracy on problems with numerators and denominators of five or less was 32% among typically achieving 6^{th} graders and 60% among typically achieving 8^{th} graders (Siegler, Thompson, & Schneider, 2011).

Children with mathematics difficulties (MD) lag behind typically achieving (TA) children in numerous aspects of fraction knowledge, including comparing and ordering fractions, estimating fraction magnitudes on a number line, performing fraction arithmetic calculations, and solving word problems involving fractions (Bailey et al., 2015; Cawley, Parmer, Yan, & Miller, 1996; Hecht & Vagi, 2010; Mazzocco & Devlin, 2008; Siegler & Pyke, 2013). In this article, we address several questions regarding learning, or all too often non-learning, of fractions. Why are fractions so hard for students in general? Why do children with MD lag behind age peers? What can be done to improve children's fraction knowledge, especially the knowledge of children with MD?

We address these questions in three sections. The first describes the integrated theory of numerical development (Siegler & Lortie-Forgues, 2014; Siegler, Thompson, & Schneider, 2011), which specifies the difficulties facing all children as they try to learn fractions. The next section reviews two recent studies by our research group (Bailey et al., 2015; Siegler & Pyke, 2013), that provide a nuanced description of developmental and individual differences in fraction knowledge among children with MD. The third section examines interventions that improve fraction knowledge of children with and without MD and discusses implications of findings from those studies for improving classroom instruction.

The Integrated Theory of Numerical Development

Although prior theories of numerical development (Geary, 2006; Gelman & Williams, 1998; Ni & Zhou, 2005; Wynn, 2002) emphasized differences between whole numbers and fractions, Siegler, Thompson, and Schneider's (2011) integrated theory of numerical development noted that numerical development involves learning about the characteristics that unite all types of real numbers as well as the characteristics that differentiate them. The key similarities noted within this theory are that all real numbers represent numerical magnitudes and that all can be represented on number lines. In addition to understanding these similarities, children also need to learn about the differences that distinguish various types of numbers. These differences include that each magnitude of a whole number is represented by a unique whole number symbol within a given symbol system (e.g., "6" in the Arabic numeral system), but that each magnitude of a decimal or a fraction are not (e.g., "6.0, 6.00, 6.000… ", "6/1, 12/2, 18/3…"); that whole numbers have unique successors and predecessors but rational numbers do not; that multiplying natural numbers always yields a product as great or greater than either operand but that multiplying decimals or fractions greater than 0 and less than 1 never does; and so on.

Within this theory, acquisition of fraction knowledge is crucial to numerical development in general, because fractions provide the first opportunity that most children have to understand that many properties of whole numbers are not properties of all numbers. The importance of understanding fraction magnitudes, especially using number lines to represent magnitudes, is emphasized by the CCSSI (2010), because its authors also viewed understanding fraction magnitudes as fundamental to understanding fraction arithmetic and mathematics more generally.

The integrated theory of numerical development also highlights the importance of understanding magnitudes for promoting arithmetic skill and mathematics achievement more generally. Accurate magnitude knowledge can help students evaluate the plausibility of answers to arithmetic problems and reject procedures that lead to implausible answers (e.g., 1/2 + 1/2 = 2/4). Consistent with this view, whole number magnitude knowledge correlates substantially with whole number arithmetic skills and with success on standardized mathematics tests (Booth & Siegler, 2006; Fazio, Bailey, Thompson, & Siegler, 2014; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Holloway & Ansari, 2009; Jordan et al., 2013; Siegler & Ramani, 2009; Vanbinst, Ghesquière, & De Smedt, 2012).

Similar relations of children's fraction magnitude knowledge to proficiency with all four arithmetic operations involving fractions and standardized math achievement test scores have been documented (Hecht & Vagi, 2010; Jordan et al., 2013; Siegler & Pyke, 2013; Siegler, Thompson, & Schneider, 2011). The relations are present not only in the U.S. but also in

European and Asian countries (Torbeyns, Schneider, Xin, & Siegler, 2014). Earlier fraction knowledge is also predictive of later knowledge of algebra and more advanced mathematics (Booth, Newton, & Twiss-Garrity, 2013; Siegler, et al., 2012). In summary, the integrated theory of numerical development proposes that fraction magnitude knowledge plays a central role in learning fraction arithmetic and contributes to learning more advanced mathematics as well.

Fraction Learning Among Children with MD

Criteria for defining MD vary greatly among investigators (see Shalev, 2007, for a review of definitions). Following the precedent of a number of other investigators (Fuchs, Fuchs, & Prentice, 2004; Hanich, Jordan, Kaplan, & Dick, 2001; Jordan, Hanich, & Kaplan, 2003), we adopted the criterion for MD of math achievement test scores in the bottom 35% of the standardization sample (for justifications of this criterion, see Gersten, Jordan, & Flojo, 2005).

In one study that used this criterion, Siegler and Pyke (2013) examined the fraction knowledge of 6th and 8th grade MD and TA children. Participants were presented three tasks measuring fraction magnitude knowledge: a 0-1 number line estimation task, in which children estimated the magnitudes of fractions on a 0-1 number line; a 0-5 number line task, in which children estimated the magnitudes of fractions on a 0-5 number line; and a magnitude comparison task, in which children chose the larger of two fractions between zero and one. Children also were presented a fraction arithmetic task that included addition, subtraction, multiplication, and division items with some problems for each operation having equal denominators and other items having non-equal denominators.

As expected, children with MD were less accurate and used less sophisticated strategies than TA children on all tasks assessing fraction magnitude knowledge. One particular problem of the children with MD was that they more often based their estimates on numerators only or denominators only. Children who did this were less accurate on 0-1 and 0-5 number line estimation as well as on 0-1 magnitude comparison problems (the only magnitude comparison problems that were presented). A likely reason was that basing estimates on numerators alone or denominators alone reflects a lack of understanding that a fraction expresses a relation between the two.

Children with MD also solved fewer fraction arithmetic problems than the TA children and used less advanced computational strategies. For example, the children with MD more often used the strategy of adding numerators and denominators separately (e.g., 3/5+2/3 = 5/8). Use of this *independent whole number strategy* always leads to incorrect answers on addition and subtraction problems. Insights into why children with MD used this strategy so often came from examining when they used it. For addition and subtraction, children with MD used the independent whole number strategy twice as often on problems with unequal as with equal denominators. This difference seems likely to reflect the children with MD using the independent whole number strategy on these problems not because they thought it was correct but rather because they did not know how to generate equal denominators while maintaining the value of the fractions. Consistent with this interpretation, frequency of use of the strategy by TA children, who were much better at generating equal denominators, was unaffected by equality of the denominator.

Another incorrect strategy, the *wrong fraction operation strategy*, was also more frequent among children with MD than among TA children. It involved inappropriately transplanting components of one fraction arithmetic operation into another, for example, transplanting a correct component of the fraction addition procedure – performing the operation on the numerators but maintaining the common denominator - into a multiplication procedure where it is incorrect (e.g., incorrectly generalizing that because 3/5 + 3/5 = 6/5, therefore 3/5 × 3/5 = 9/5).

Another interesting and somewhat surprising finding was that many children knew and used both correct and incorrect strategies on virtually identical problems (e.g., addition problems with equal denominators, such as 3/5 + 2/5 and 3/5 + 1/5). Both MD and TA children showed such strategic variability, but it was greater among children with MD. Sometimes children used two different incorrect strategies, but more often they used a correct and an incorrect strategy. Thus, on 64% of similar problem pairs (problems with the same fraction arithmetic operation where both operands had equal denominators or both had unequal denominators) on which children with MD used different strategies, they used one correct and one incorrect strategy. In these cases, use of the correct strategy showed that the children knew that procedure; their lack of consistent use of it suggested that they either did not know that it was correct, only inconsistently remembered it, or only sporadically attended to the problem sufficiently to use the correct procedure (Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007).

Especially disturbing, children with MD not only started middle school with less knowledge of fraction arithmetic, they made much slower progress than the TA children once they were there. For all four arithmetic operations, the TA children's accuracy increased more between Grade 6 and Grade 8 than that of the children with MD. The TA children's accuracy increased significantly on all four fraction arithmetic operations, whereas the accuracy of children with MD did not increase significantly on any of them. Thus, the MD students started middle school behind in fraction knowledge, and fell further behind during it.

Table 1: Mean Performance on Fraction Tasks by Grade and Achievement Level

Study	**Task**	**Grade**	**US TA**	**US MD**	**China TA**	**China MD**
Bailey et al., 2015	Arithmetic (% correct)	6	52	15	100	76
		8	80***	20	100	78
	0-1 Number Line (PAE)	6	10	26	5	20
		8	4***	24	5	16
	0-5 Number Line (PAE)	6	23	41	10	27
		8	11***	36	7***	20*
	Magnitude Comparison (% correct)	6	81	42	88	48
		8	93**	57*	93*	61
Siegler & Pyke, 2013	Arithmetic (% correct)	6	49	33		
		8	73***	40		
	0-1 Number Line (PAE)	6	11	25		
		8	7*	22		
	0-5 Number Line (PAE)	6	19	32		
		8	14	26		
	Magnitude Comparison (% correct)	6	80	58		
		8	83	68		

Note. TA = typically achieving children; MD = children with math difficulties; PAE = *Percent Absolute Error*. Asterisks denote improvement between sixth and eighth grade; * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$.

In a second study, Bailey et al. (2015) compared the fraction knowledge of American and Chinese 6th and 8th graders on the same fraction tasks as used by Siegler and Pyke (2013). In both samples, children scoring in the top 2/3 of performance on each task within their society were classified as TA, and classmates scoring in the bottom 1/3 were classified as MD.

The data indicated that being in the bottom 1/3 of the distribution in one's country does not imply that performance must be poor in absolute terms. For fraction arithmetic, Chinese 6th graders who scored in the bottom 1/3 of their country's distribution were as accurate as the U.S. TA 8th graders. The data on number line estimation were similar but less strong. Estimation accuracy on the number line task was measured as *percent absolute error* (PAE, defined as *(|Estimate – Correct Answer|/Numerical Range)*100%.*) For example, if a student was asked to estimate 47 on a 0-100 number line, and the student marked a location corresponding to 27, the PAE would be 20%. As shown in Table 1, Chinese children estimated the location of numbers on the number line more accurately, regardless of whether they were in the top 2/3 or the bottom 1/3 of children within their country on the measure.

Note that the MD/TA distinction was made on the basis of relative performance within the child's country. In absolute terms, fraction arithmetic accuracy of Chinese children in the bottom 1/3 of their country's distribution was reasonably good – 76% correct in 6th grade and 78% correct in 8th grade. Although the present study did not include measures of general cognitive abilities, prior studies indicate small or nonexistent differences in cognitive abilities between U.S. and Chinese children (Stevenson & Stigler, 1992), which makes it unlikely that differences in fraction arithmetic accuracy among U.S. and Chinese children with MD were attributable to such general cognitive differences. The present findings indicate that children near the bottom of their country's distribution do not necessarily possess poor fraction understanding. In particular, it suggests that, with better instruction and greater time spent practicing mathematics, American children in the bottom 35% of the distribution are capable of producing considerably better fraction performance than they currently do.

In summary, the Siegler and Pyke (2013) and Bailey et al. (2015) studies depicted in detail the difficulties in fraction learning facing children with MD, both in understanding magnitudes and in mastering arithmetic computation. U.S. children with MD showed a disturbing lack of progress between 6th and 8th grade in arithmetic accuracy. On a more optimistic note, the data of Chinese children demonstrated that being toward the bottom of one's country's distribution of mathematics proficiency does not doom a child to low levels of performance in absolute terms. In China, even children in the bottom 1/3 of performance within their country showed reasonably good knowledge of both fraction magnitudes and fraction arithmetic. The finding suggests that high quality teaching and substantial practice allows even children toward the bottom of the distribution to acquire reasonably good fractions knowledge.

Interventions for Improving Fraction Knowledge

Given the importance of fractions and decimals for learning mathematics and many students' poor understanding of them, it is not surprising that many interventions have attempted to improve learning of them. A common feature of the most successful interventions is that they help children understand how these numbers map onto magnitudes by using the number line representation (Fujimura, 2001; Moss & Case, 1999; Fuchs, et al., 2013; 2014; in press a & b; Rittle-Johnson, Siegler, & Alibali, 2001; Schneider, Grabner, & Paetsch, 2009).

Compared to the traditional part-whole interpretation of fractions, number lines seem to be a more useful tool for teaching. One advantage is that number lines reduce the difficulty of introducing improper fractions and suggest their continuity with other numbers. Another advantage is that the continuity of number lines implies that there are an infinite number of fractions between any two numbers.

Consistent with this analysis, a curriculum unit that utilized the number line representation successfully improved fraction understanding (Saxe, Diakow, & Gearhart, 2012). During the intervention, 4th and 5th graders first received instruction on whole numbers with the number line as the principal representation. Then, the children learned fractions within the same number line context that they already knew for whole numbers.

Although the 19 intervention lessons were not taught in succession, and the children in the intervention group followed the same curriculum as children in the comparison group for the rest of their classes, the intervention produced impressive gains. Compared with children in the comparison group, children who received the intervention scored higher on end-of-unit tests immediately after the intervention and on end-of-year tests five months after the intervention that assessed knowledge of whole numbers and fractions. This advantage was evident on problems involving number lines and on other problems as well. Especially encouraging, the greater learning gains of children in the intervention group were present regardless of the children's initial mathematical knowledge. In fact, children with MD (defined as those in the bottom one-third of the distribution) in the intervention group performed similarly to medium-achievers (those in the middle one-third of the distribution) in the comparison group in the end-of-year tests.

Fuchs and colleagues (Fuchs et al., 2013, 2014; in press a & b) also emphasized number lines in a series of intervention studies and produced impressive improvements in fraction knowledge of fourth graders with MD (see Fuchs' article in this issue for more detailed descriptions of these studies). Especially striking, gains in understanding of fraction magnitudes mediated the intervention effects – statistically controlling for changes in magnitude knowledge reduced or eliminated effects of participation in the intervention condition. These mediation effects suggest that understanding of fraction magnitudes is an essential component of fraction learning.

Discussion and Conclusions

The integrated theory of numerical development helps explain differences in fraction knowledge between TA children and peers with MD. The theory posits that the development of fraction magnitude knowledge is a central contributor to fraction learning and to mathematics achievement more generally (Siegler, Thompson, & Schneider, 2011). This view suggests that the fraction competence of children with MD can be improved by developing their fraction magnitude knowledge. Indeed, several interventions with number lines demonstrate that it is possible to improve fraction magnitude knowledge of children with MD and that such interventions generalize to enhancing learning of fraction arithmetic (Fuchs, et al., 2013; 2014; in press a & b; Saxe, Diakow, & Gearhart, 2012).

Although it is no surprise that children with MD know less about fractions than TA children, several other findings were unexpected. For example, the achievement gap between U.S. MD and TA children, already present in elementary school, becomes considerably larger in middle school. In our studies, American children with MD did not show significant gains between 6th and 8th grade of 0-1 or 0-5 number line estimation nor on any of the four arithmetic operations

(Bailey et al., 2015; Siegler & Pyke, 2013). In contrast, the TA children in the U.S. made considerable progress in the same period of time on almost all fraction tasks (see Table 1). A similar pattern was found in a prior short-term longitudinal study of the period between Grade 4 and Grade 5 (Hecht & Vagi, 2010). There too, the fraction knowledge of children with MD improved less from one grade to the next than that of TA peers.

A hopeful finding from Bailey et al. (2015) was that the poor performance on fraction tasks among the American children with MD did not occur among Chinese children in the same part of their country's distribution of achievement test scores. The reason we see this finding as hopeful is that it indicates that being in the bottom one-third of one's country's distribution of mathematics performance does not doom children to having poor fractions knowledge. With high quality teaching and extensive practice, children with MD in the U.S. might reach similar levels of mastery of fractions.

Another hopeful finding was that interventions, have greatly improved the fraction knowledge of U.S. children with MD. These successful interventions put great emphasis on representing fraction magnitudes with number lines, a practice that was also recommended in CCSSI (2010). Greater use of number lines in the classroom thus has the potential to help children with MD better understand fraction magnitudes and fraction arithmetic.

Acknowledgments
We would like to thank Drew H. Bailey for his insightful comments on an earlier version of this manuscript. The opinions expressed are those of the authors and do not represent views of the Institute or the U.S. Department of Education.

Declaration of Conflicting Interests
The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding
The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: The research reported here was supported in part by the Institute of Education Sciences, U.S. Department of Education, through Grants R305A150262 and R324C100004:84.324C, Subaward 23149 to Carnegie Mellon University, in addition to the Teresa Heinz Chair at Carnegie Mellon University and the Siegler Center of Innovative Learning, Beijing Normal University.

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