Braithwaite, D. W., & Siegler, R. S. (2018). Developmental changes in whole number bias. *Developmental Science, 21*(2), e12541. doi: 10.1111/desc.12541.

# Developmental Changes in the Whole Number Bias

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Many students' knowledge of fractions is adversely affected by whole number bias, the tendency to focus on the separate whole number components (numerator and denominator) of a fraction rather than on the fraction's magnitude (ratio of numerator to denominator). Although whole number bias appears early in the fraction learning process and under speeded conditions persists into adulthood, even among mathematicians, little is known about its development. Performance with equivalent fractions indicated that between fourth and eighth grade, whole number bias decreased, and reliance on fraction magnitudes increased. These trends were present on both fraction magnitude comparison and number line estimation. However, analyses of individual children's performance indicated that a substantial minority of fourth graders did not show whole number bias and that a substantial minority of eighth graders did show it. Implications of the findings for development of understanding of fraction equivalence and for theories of numerical development are discussed.

Understanding fractions is crucial for success in mathematics, science, and many occupations (Booth, Newton, & Twiss-Garrity, 2014; McCloskey, 2007; Siegler et al., 2012). Unfortunately, children in the U.S. and many European countries experience great difficulty gaining this understanding (Torbeyns, Schneider, Xin, & Siegler, 2015). The problem often persists into adulthood, for example among community college students (Fazio, DeWolf, & Siegler, 2016; Schneider & Siegler, 2010; Stigler, Givvin, & Thompson, 2010).

A major obstacle to understanding fractions is whole number bias – the tendency to focus on the whole number components of fractions (numerators and denominators) rather than thinking of a fraction as a single number (Ni & Zhou, 2005). Reflecting whole number bias, children often add or subtract fractions by adding or subtracting both their numerators and denominators (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980; Siegler, Thompson, & Schneider, 2011). For example, children asked to solve 1/8+1/8 often proposed the answer 2/16 (Mack, 1995).

Whole number bias also interferes with children's understanding of the magnitudes of fractions by creating a misperception that fractions with larger whole number components have larger magnitudes. This misperception is reflected in higher error rates and longer response times on fraction magnitude comparison tasks when the larger fraction has smaller components (DeWolf & Vosniadou, 2014; Fazio et al., 2016; Meert, Grégoire, & Noël, 2010a, 2010b). For example, participants are often slower and less accurate at recognizing that 2/5 is larger than 3/9 than at recognizing that 2/5 is smaller than 4/9, despite the distance between the magnitudes of the latter pair of fractions being smaller.

In the present study, we focus on the effects of whole number bias on understanding of fraction magnitudes; we use the term "whole number bias" to refer specifically to such effects. This focus is justified by the strong relations between fraction magnitude understanding and success in more advanced mathematics (Siegler & Braithwaite, 2016). Fraction magnitude understanding is both correlated with and predictive of later proficiency in fraction arithmetic and overall mathematics achievement, even after controlling for plausible third variables, including academic achievement in other areas such as whole number arithmetic, reading, executive function, and non-symbolic numerical knowledge (Booth et al., 2014; Fazio, Bailey, Thompson, & Siegler, 2014; Siegler & Pyke, 2013; Siegler et al., 2012). Moreover, randomized controlled trial interventions designed to improve fraction magnitude understanding also lead to improvements in fraction arithmetic and varied measures of conceptual understanding of fractions (Fuchs et al., 2013, 2014, in press).

Despite the importance of the whole number bias, little is known about its development. Alibali and Sidney (2015) noted that "the bias is evident, not only in learners who have just been introduced to rational numbers, but also in individuals who have extensive familiarity with rational numbers," including adults. This persistence raises the question: Does whole number bias decrease over the course of development?

As we argue below, existing evidence is inconclusive with respect to whether whole number bias decreases at all. Further, even if whole number bias does decrease, neither the timing and extent of this decrease, nor the mechanisms underlying it, are well understood. The present study addressed these questions by tracking the developmental trajectory of the whole number bias from fourth to eighth grade, the period in which fractions, ratios, and proportions receive the greatest instructional attention (Common Core State Standards Initiative, 2010). To understand the mechanisms underlying developmental changes in whole number bias, we tracked changes over the same time period in the distribution of different types of fraction representation among individual children.

**Whole Number Bias**

Existing evidence for effects of whole number bias on fraction magnitude understanding comes mainly from fraction magnitude comparison tasks (for exceptions, see Bright, Behr, Post, & Wachsmuth, 1988; Kerslake, 1986; and Ni, 2001). Performance on magnitude comparison tasks improves with age during primary and middle school. For example, between fifth and seventh grade, fraction magnitude comparison accuracy improved from 75% to 90% in Meert et al. (2010b) and from 68% to 94% in Gabriel et al. (2013). These improvements could reflect decreasing effects of whole number bias on children's understanding of fraction magnitudes.

Alternatively, however, changes in magnitude comparison accuracy could reflect changes in strategy use. Despite the name of the task, fraction magnitude comparison can be performed using a variety of strategies that do not involve fraction magnitudes at all. For example, two fractions can be compared by simply judging the fraction with the larger numerator to be larger. Although this strategy yields correct answers for many comparisons, using it in all cases leads to errors consistent with whole number bias, such as the incorrect judgment that 3/9 is larger than 2/5. However, the strategy does not involve fraction magnitudes, only numerator magnitudes, so its use does not imply biased representations of fraction magnitudes. Similarly, when one denominator is equal to the other denominator multiplied by a whole number N, people can multiply the fraction with the smaller denominator by N/N and then judge the fraction with the larger numerator as larger. For example, one can compare 2/3 and 4/9 by multiplying 2/3 by 3/3 to obtain 6/9, and then observe that 6>4. Such a strategy can reflect consideration of fraction magnitudes, but it also can be used mechanically without considering either fraction's magnitude.

These are not just logical possibilities; both children and adults often use such strategies on fraction magnitude comparison problems (Bonato, Fabbri, & Umiltà, 2007; Fazio et al., 2016). For example, in Fazio et al. (2016), individual students at both a highly selective university and a non-selective community college averaged 10-11 distinct strategies in solving a set of 48 fraction magnitude comparison problems. The most common strategies involved component-wise comparison without reference to fraction magnitudes.

To avoid use of strategies specific to the magnitude comparison task that allow comparison without reference to fraction magnitudes, the present study assessed whole number bias using number line estimation, in which participants estimate the magnitudes of fractions by placing them on a number line (Bright et al., 1988; Fazio et al., 2014; Iuculano & Butterworth, 2011; Kerslake, 1986; Meert, Grégoire, Seron, & Noël, 2012; Opfer & Devries, 2008; Resnick et al., 2016; Thompson & Opfer, 2008). Number line estimates are generated for individual fractions in isolation, and thus are not subject to the componential comparison strategies described above. Participants were also presented a fraction magnitude comparison task to provide continuity with previous studies. The two tasks together promised to provide a more accurate depiction of developmental changes in whole number bias than either alone could.

**Representations of Fractions**

Another goal of the present study was to understand what type of representations give rise to whole number bias, and how changes in these representations contribute to changes in whole number bias. According to one account, whole number bias results from reliance on componential representations – that is, representations that reflect the sizes of fractions' whole number components, that is, numerator and denominator (Bonato et al., 2007). Componential representations do not directly reflect the integrated magnitudes of fractions – that is, the ratio of numerator to denominator – at all. Consistent with componential representations, in a fraction comparison task, the distance between two fractions' whole number components, but not the distance between their integrated magnitudes, predicted response times (Bonato et al., 2007). An alternate account proposes hybrid representations that reflect influences of both component sizes and integrated fraction magnitudes. Supporting this account, several studies have found effects of both componential distance and difference in integrated magnitudes on fraction magnitude comparisons (Meert, Grégoire, & Noël, 2009; Meert et al., 2010a, 2010b; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013).

Evidence for hybrid representations is not conclusive, however, because the above-mentioned studies analyzed data aggregated across participants. This fact leaves open the possibility that some participants relied entirely on componential representations and others relied entirely on fraction magnitudes, creating the appearance of hybrid representations without any individual relying on such representations. In the area of numerical cognition, differences between group and individual data patterns are quite common. For example, Siegler (1989) found that a model of children's subtraction that fit the aggregated data quite well did not fit any individual child. Similarly, Siegler (1987) found that a model of children's addition that fit the aggregated data very well was used on only about one-third of trials.

Thus, it remains an open question whether whole number bias in children's representations of fractions results from the use of hybrid representations by most or all individuals, use of componential representations by some individuals and fraction magnitude representations by others, or some combination of all three forms of representation. Although we made no specific prediction regarding the absolute frequencies of these models, we expected decreases in the frequencies of componential representations, hybrid representations, or both to accompany any developmental decreases in the whole number bias, with reliance on fraction magnitudes becoming more common with decreases in whole number bias.

**Equivalent Fractions**

Equivalent fractions were well suited to examining the above issues. Fractions are equivalent if their components (numerators and denominators) stand in the same ratio (e.g., 4/5 = 16/20). Thus, equivalent fractions have the same magnitude and are interchangeable in tasks relating to magnitude, including magnitude comparison and number line estimation. However, based on prior research regarding whole number bias, we expected that many children would not treat equivalent fractions as having equal magnitudes, but would instead treat fractions with larger components (e.g., 16/20) as larger than equivalent fractions with smaller components (e.g., 4/5).

Including equivalent fractions in the present study had the important advantage of permitting manipulation of component size completely independently of fraction magnitude. That is, for a given magnitude, we could include fractions of different component sizes but identical magnitudes (e.g., 4/5 and 16/20).

Equivalent fractions are also an important subject in their own right. Fraction equivalence is at the heart of a crucial difference between fractions and whole numbers: Each whole number has a unique representation using Arabic numerals (e.g., "6") or spoken language (e.g., "six"), but each fraction can be expressed in infinitely many ways (e.g., 4/5, 8/10, 12/15, 16/20…). Thus, understanding fraction equivalence increases understanding of which properties of whole numbers are not true of numbers in general – a central theme in numerical development (Siegler et al., 2011; Smith, Solomon, & Carey, 2005; Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2010).

Moreover, understanding fraction equivalence is essential to understanding fraction arithmetic procedures. For example, to understand the standard procedures for adding and subtracting fractions, learners must know that substituting equivalent fractions does not change the magnitude of the operands and therefore does not change the answer (e.g. 4/5 + 1/4 = 16/20 + 5/20 = 21/20). If 4/5 were not equivalent to 16/20, or if 1/4 were not equivalent to 5/20, this substitution and the ensuing answer would have no logical basis.

Previous work has examined understanding of fraction equivalence using fraction-model conversion tasks and purely symbolic conversion tasks. In fraction-model conversion tasks that are used to assess understanding of equivalent fractions, children represent a fraction using a graphical model partitioned according to a different denominator – for example, by marking 5/3 on a number line segmented into 12ths. Children often perform poorly on such tasks (Kamii & Clark, 1995; Ni, 2001), even if they perform correctly purely symbolic conversions such as converting 5/3 into 20/12 (Bright et al., 1988). On the other hand, use of pre-segmented graphical models in fraction-model conversion tasks with equivalent fractions may increase children's use of incorrect counting strategies (Boyer, Levine, & Huttenlocher, 2008), inhibiting performance even among children whose internal representations of equivalent fractions are actually equal. Thus, fraction-model conversion tasks may under-estimate children's understanding of equivalence.

In the present study, we assessed whether children treat equivalent fractions as equal in magnitude comparison and number line estimation tasks when the equivalent fractions are presented on different trials. These tasks avoid the limitations of the above conversion tasks because they do not explicitly involve conversion between equivalent fractions. Of course, this strength is also a weakness, in that our tasks do not directly assess whether equivalent fractions are explicitly recognized as equal. No task is perfect; we hoped that the measures employed in the present study would offer advantages complementary to those of previous studies using fraction conversion tasks. An additional advantage of using the number line estimation task to assess understanding of fraction equivalence is that this task aligns well with the Common Core State Standards Initiative (2010), which states that children should "understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line."

## Experiment 1

Experiment 1 included two tasks that were designed to test for the predicted effects of whole number bias: a fraction number line estimation task and a fraction magnitude comparison task. On each trial of the number line estimation task, children placed a fraction drawn from an equivalent pair onto a 0-1 number line. Fractions with larger components were expected to elicit larger estimated magnitudes than equivalent fractions with smaller components.

On each trial of the magnitude comparison task, one of the two fractions from an equivalent pair was compared to a larger comparison fraction, with the particular comparison fraction varying from trial to trial. Children were expected to judge fractions with larger components as larger than the comparison fraction more frequently than equivalent fractions with smaller components.

Fourth and fifth graders were chosen to test the strength of whole number bias effects on these tasks. Although the age/grade difference was only one year, the substantial fractions instruction that children receive during this year made it plausible that the strength of the whole number bias would decrease during this period.

### Method

**Participants.** Participants included 66 children, 33 fourth graders (mean age=9.8 years) and 33 fifth graders (mean age=10.9 years), 30 males and 36 females, all attending an elementary school near Pittsburgh, PA. In this school, 64% of students received free or reduced price lunches. Mathematics achievement at the school was below average for the state in which the study was conducted. On the mathematics portion of the Pennsylvania System of School Assessment (PSSA), the standardized achievement test used in Pennsylvania, 42% of fourth graders and 74% of fifth graders scored below the basic level, compared to 25% and 26% respectively statewide. The student body was 74% African-American, 12% multiracial, 11% Caucasian, and 4% "other." Two female Caucasian research assistants administered the experiment, which was conducted near the end of the school year.

**Materials.** Two sets of stimuli were created. Each set consisted of 13 pairs of equivalent fractions, for a total of 26 fractions in each set. Numerators within each set ranged from 1 to 24 and denominators from 2 to 28; each set contained one pair of fractions with magnitudes equal to 0.5 and three pairs of fractions with magnitudes in each quadrant of the range from 0 to 1 (excluding 0.5). Each equivalent pair included one fraction with a single digit denominator (e.g. 4/5) and one with a two-digit denominator (e.g. 16/20); these will be referred to as small component fractions and large component fractions, respectively. The small component fractions were in lowest terms, with one exception[1]. All fractions are listed in the Supporting Information, as are the instructions that children were given.

**Procedure.** The study was conducted in a whole class format during the children's mathematics class period. Within each grade, 17 children were presented with one of the two fraction sets and 16 children with the other. Children were not told that the tasks involved equivalent fractions, and successive problems never involved fractions from an equivalent pair. The tasks were presented in a printed packet, which children completed working individually at their own pace, without any stated time limit. The number line estimation task was always presented first, followed by the magnitude comparison task.

On the number line estimation task, each fraction was presented above a 0-1 line, one item per page. Children were instructed to mark each fraction's position on the number line. The fractions were presented in a fixed random order or its reverse, with half of the children receiving each order. To ensure that all of each child's data for small and large component fractions represented fractions of equal magnitude, equivalent fraction pairs were excluded if either fraction was left blank. This filter resulted in exclusion of 5.9% of trials, including all trials from 2 children, leaving 64 children for analysis. Among those children, 3.0% of trials were excluded.

On the magnitude comparison task, a fraction with components intermediate in size relative to those in each equivalent pair was chosen as the comparison fraction for both equivalent fractions in the pair. For example, the comparison fraction for the pair "4/5, 16/20" was 10/11. Each of the equivalent fractions was presented once together with the comparison fraction for the pair; children were instructed to circle the larger of the two fractions in that comparison. Different fractions were used as the comparison fractions for different equivalent pairs.

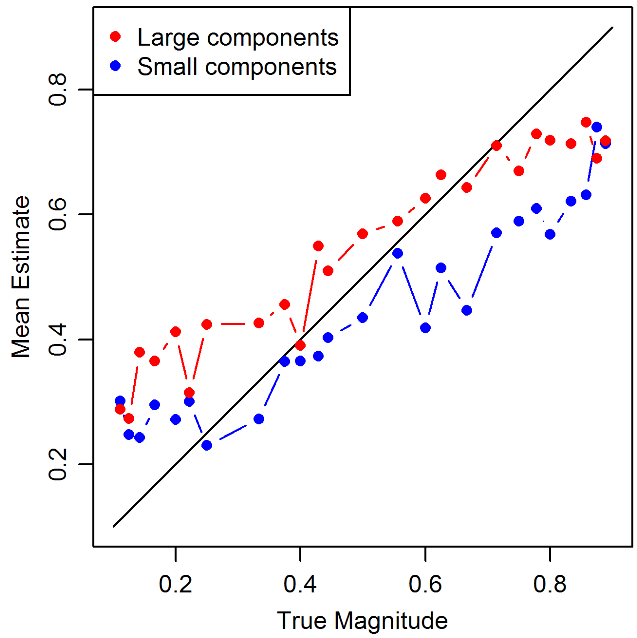
The equivalent fractions were always smaller than their comparison fractions (by at least 0.05, mean=0.116), so the comparison fraction was always the correct answer. Because the comparison fractions were not marked as such, there was no obvious way for children to know which numbers were considered comparison fractions for our purposes. Equivalent fractions and comparison fractions each appeared equally often on the left and right side of the page. Problems were presented either in a fixed random order or the reverse of that order. Data were filtered in the same way as for number line estimation, resulting in the exclusion of 3.7% of trials, including all trials from 1 child, leaving 65 children for analysis. For those children, the percent of trials excluded was 2.2%.

## Results

**Number Line Estimation.** To assess children's competence at the number line estimation task, and to ensure comparability with previous samples, Percent Absolute Error (PAE), the absolute value of the difference between estimated and true value, was calculated on each trial. Mean PAE was 19.3% and did not differ between fourth and fifth graders; the PAE was similar to that in prior studies with this age group (e.g., Fazio et al., 2014).

Next, to assess whole number bias, the mean value of each child's estimates was calculated separately for small and large component fractions, and these data were submitted to an ANOVA with component size as a within-subject variable, and grade and fraction set as between-subjects variables. This analysis yielded a single main effect for component size, $F$(1, 60) = 27.37, $p$ < .001, $\eta_g^2$ = .130, and no interactions. Consistent with the whole number bias, on average, children's number line estimates for large component fractions were 0.104 larger than for equivalent small component fractions. Estimates of 67% of children (43 of 64) were larger for large than for small component fractions, $\chi^2$(1, N = 64) = 7.56, $p$ = .006.

*Figure 1.* Mean estimates for large and small component fractions plotted against true magnitudes (Experiment 1). The 45° diagonal indicates the locations of normatively correct estimates.

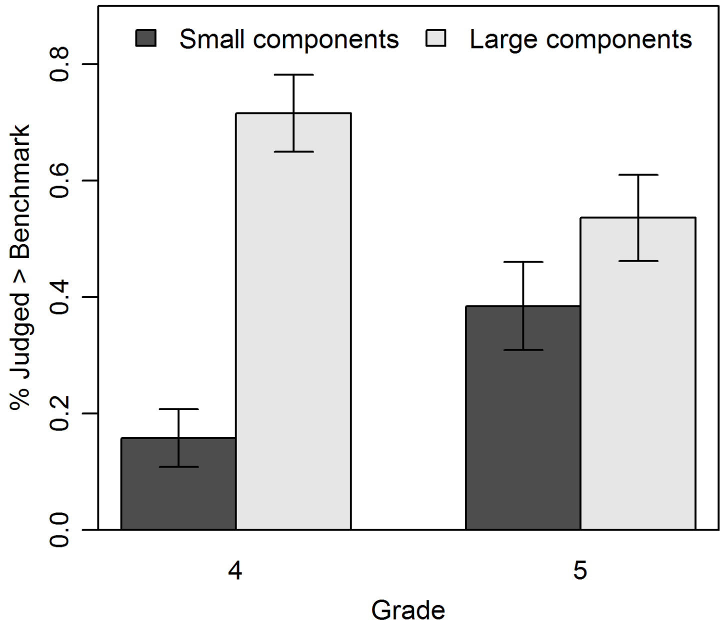



An items analysis, in which the mean estimate for each fraction was calculated across subjects, was conducted to better understand the effect of component size on number line estimates. These data, shown in Figure 1, were submitted to an ANOVA, treating each equivalent pair as an item and component size (larger or smaller) as a within-items variable. The effect of component size was significant, $F(1, 25) = 52.47$, $p < .001$, $\eta_g^2 = .102$. For 24 of 26 equivalent pairs, mean estimates were larger for the fraction with larger components, $\chi^2(1, N = 26) = 18.6$, $p < .001$.

**Magnitude Comparison.** Children correctly answered 55.2% of magnitude comparison items, which was greater than chance (50.0%), $t(64) = 3.64$, $p < .001$. Accuracy did not differ between fourth and fifth graders, 56.3% versus 54.0%.

An ANOVA with component size as a within-subject variable, and grade and fraction set as between-subjects variables, revealed that fractions with numerators and denominators larger than those of the comparison fractions were judged to be larger than the comparison fractions more often than equivalent fractions with components smaller than those of the comparison fractions were (62.7% versus 27.0%), $F(1, 61) = 15.6$, $p < .001$, $\eta_g^2 = .189$. Almost 70% of children (45 of 65) more often judged as larger equivalent fractions with components larger than those of the comparison fractions than equivalent fractions with components smaller than those of the comparison fractions, $\chi^2(1, N = 65) = 9.62$, $p = .002$.

*Figure 2.* Percent of equivalent fractions judged to be larger than corresponding comparison fractions in Experiment 1. Error bars indicate standard errors.




Component size also interacted with grade, $F(1, 61) = 5.02$, $p = .029$, $\eta_g^2 = .070$ (Figure 2). Paired *t*-tests indicated that the effect of component size on magnitude comparison judgments was significant in fourth grade, $t(32) = 5.02$, $p < .001$, but not in fifth grade, $t(31) = 1.06$, $p = .300$. The proportion of children who judged large component fractions to be larger more often than equivalent small component fractions decreased from 79% of fourth graders to 59% of fifth graders, though this difference was not significant, $\chi^2(1, N = 65) = 2.04$, $p = .154$.

An items analysis, in which each equivalent pair was treated as an item and component size as a factor in a repeated-measures ANOVA, yielded an effect of component size, $F(1, 25) = 699.9$, $p < .001$, $\eta_g^2 = .910$. In all 26 equivalent pairs, the fraction with components larger than those of its comparison fraction was judged to be larger more often than the equivalent fraction with components smaller than those of that comparison fraction.

**Discussion**

Results of Experiment 1 revealed large and consistent effects of component size on children's responses to equivalent fractions, thus providing unambiguous evidence of whole number bias. The results replicated previous findings from magnitude comparison tasks with non-equivalent fractions (e.g. Meert et al., 2010b). The findings also demonstrated for the first time that analogous effects appear in number line estimation, where many strategies that can be used on fraction magnitude comparison tasks without considering magnitudes are inapplicable. Thus, whole number bias is not merely an artifact of strategy choices on magnitude comparisons; instead, this bias appears to reflect a more general feature of children's representations of fractions[2].

The effect of component size on magnitude comparison decreased from fourth to fifth grade, but no decrease was seen with number line estimation. Whole number bias in fraction number line estimation might not decrease with age, or it might decrease over a longer period than that examined in Experiment 1. To test these possibilities, Experiment 2 included children ranging from fourth to eighth grade.

## Experiment 2

The first goal of Experiment 2 was to evaluate the degree to which whole number bias decreases from fourth to eighth grade. We expected whole number bias to decrease considerably over this period, because of the substantial instruction that children receive in fractions during it and because prior findings indicate that children's knowledge of fraction magnitudes increases during it (Resnick et al., 2016; Siegler et al., 2011).

A second goal was to determine what changes, if any, occur in children's representations of fractions over this period. As reviewed in the Introduction, whole number bias in data aggregated across participants could reflect (1) all children relying on hybrid representations, which reflect influences of both fraction magnitudes and component magnitudes; (2) some children relying on component magnitudes only and others on fraction magnitudes only; or (3) some children relying on hybrid representations, others on component magnitudes only, and others on fraction magnitudes only. Because each of these possibilities would create the appearance of hybrid representations at the group level, we evaluated the fit of different models of fraction representation at both the group and individual levels, using Bayesian model comparison. Analyzing the distribution of representations among different individuals also allowed us to investigate whether this distribution changes over developmental time, concurrent with the hypothesized decrease in whole number bias.

A third goal was to exclude a possible interpretation of the results of Experiment 1. Because the small component fractions in Experiment 1 were almost all in lowest terms, the findings may have reflected differing understanding of fractions that are or are not in lowest terms, rather than effects of component size as such. Textbooks and teachers appear usually to present fractions in lowest terms, so greater familiarity with fractions in that form provided a plausible explanation for the Experiment 1 results. Experiment 2 included multiple equivalent fractions for each magnitude, allowing us to observe effects of component size among equivalent fractions, none of which were in lowest terms.

### Method

**Participants.** Participants included 137 children; 46 fourth graders (mean age=9.2 years), 49 sixth graders (mean age=11.1 years), and 42 eighth graders (mean age=13.1 years); 62 males and 75 females. All attended an elementary school near Pittsburgh, PA, a different school from that in Experiment 1. In this school, 64% of students received free or reduced lunch. Students scored near state averages on the mathematics portion of the PSSA (the proportions of fourth, sixth, and eighth graders in the school scoring below basic level in were 33%, 21%, and 24% respectively, compared to 25%, 25%, and 38% statewide). The student body was 52% Caucasian, 47% African-American, and 1% "other." The experiment was administered by two female Caucasian research assistants and was conducted in the first half of the school year.

**Materials.** Stimuli were 11 groups of equivalent fractions with 4 fractions in each group, a total of 44 fractions. All 44 were between 0 and 1, with numerators ranging from 1 to 20 and denominators ranging from 2 to 36. Each group of equivalent fractions included one fraction that was in lowest terms; these fractions were 1/5, 2/9, 1/4, 1/3, 3/7, 1/2, 5/9, 2/3, 3/4, 4/5, and 5/6. The remaining fractions were obtained by multiplying the lowest-terms fraction in each group by three different numbers N/N, with N ranging from 2 to 15. Within each group, the lowest-terms fraction and the fraction obtained by multiplying by the smallest N/N, usually 2/2 or 3/3, were classified as small component fractions (e.g., 4/5, 8/10), while the remaining two fractions were classified as large component fractions (e.g., 12/15, 20/25). All fractions are listed in the Supporting Information, as are the instructions that children were given.

**Procedure.** The number line estimation task was conducted using the same procedure as in Experiment 1, with a few exceptions noted below. To allow time for the larger number of number line estimation trials, the magnitude comparison task was not presented in Experiment 2.

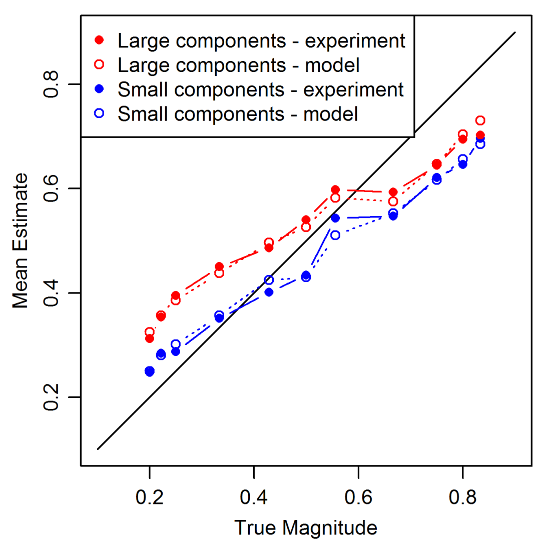
The same set of number line estimation problems was presented to all children in a fixed random order or the reverse of that order, with half of children receiving the items in each sequence. Trials were excluded if left blank. If a given child did not provide a response for at least one large and one small component fraction within a given group of equivalent fractions, then all four trials for that group were excluded. This filter resulted in exclusion of 0.8% of all trials; no participants were excluded.

## Results

**Number Line Estimation.** Mean PAE on the number line estimation task was 15.4%. Estimation accuracy improved with grade, $F(1, 135) = 5.35$, $p < .022$, $\eta_g^2 = .038$. Mean PAE was 17.9% in fourth grade, 15.5% in sixth grade, and 12.5% in eighth grade; pairwise comparisons between grades did not reach significance.

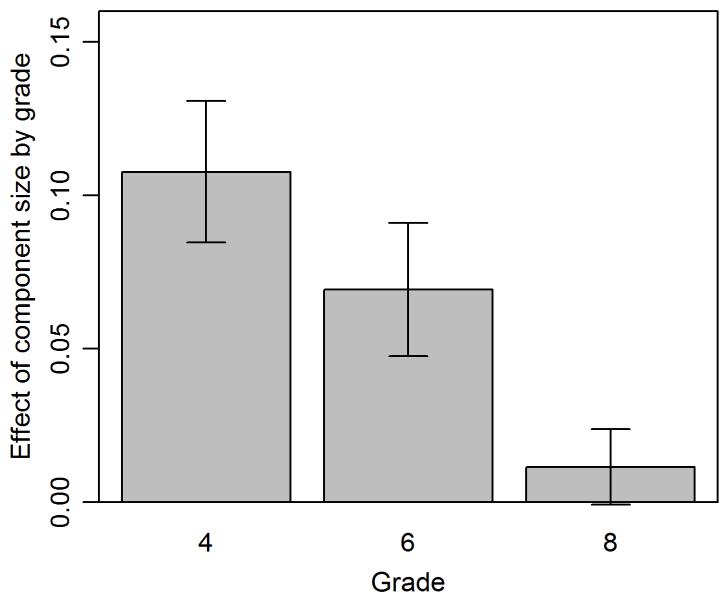
To assess whole number bias, the mean value of each child's estimates was calculated separately for the half of fractions classified as large and the half classified as small component fractions. These data were submitted to an ANOVA with component size (large versus small) as a within-subject variable and grade (fourth, sixth, or eighth) as a between-subjects variable. This analysis revealed a main effect for component size, $F(1, 135) = 31.0$, $p < .001$, $\eta_g^2 = .090$. Estimates for large component fractions averaged 0.065 larger than for equivalent small component fractions. In all 11 equivalent fraction groups, the mean estimate for the two large component fractions was larger than that for the two small component fractions (Figure 3).

*Figure 3.* Mean estimates for large and small component fractions plotted against true magnitudes (Experiment 2). Solid lines indicate experimental data; dashed lines indicate predictions of the hybrid model, as described in the text. The 45° diagonal indicates the locations of normatively correct estimates.



Critically, a component size by grade interaction was also present, $F(1, 135) = 11.0$, $p = .001$, $\eta_g^2 = .033$. The interaction reflected the difference in mean estimates between large and small component fractions decreasing with grade (Figure 4). Paired *t*-tests indicated that component size influenced estimates of fourth graders, $t(45) = 4.66$, $p < .001$, and sixth graders, $t(48) = 3.18$, $p = .003$, but not of eighth graders, $t(41) = 0.93$, $p = .357$. The proportion of children who generated larger estimates for large component fractions decreased from 67% of fourth graders to 59% of sixth graders to 50% of eighth graders, though the differences between age groups were not significant, $\chi^2(2, N = 137) = 2.75$, $p = .253$.

*Figure 4.* Effect of component size by grade level (Experiment 2), calculated as the difference in mean estimates between large and small component fractions. Error bars indicate standard errors.




Because one of the two small component fractions in each set was in lowest terms, whereas none of the large component fractions were, a difference between fractions that were or were not in lowest terms – rather than an effect of component size as such – could have caused the observed differences between small and large component fractions. To test this possibility, the above analysis was repeated with the data for fractions in lowest terms excluded. The effect of component size remained, $F(1, 135) = 24.7$, $p < .001$, $\eta_g^2 = .047$, as did the interaction of size with grade, $F(1, 135) = 13.8$, $p < .001$, $\eta_g^2 = .027$.

**Group versus Individual Analyses of Fraction Representations.** To evaluate alternate accounts of whole number bias and developmental changes therein, number line estimates were submitted to three linear regression models: a fraction magnitude model, which included fraction magnitude as the only predictor; a componential model, which included numerator and denominator as the only predictors; and a hybrid model, which included all three of these predictors. Bayes Factors were calculated for each model relative to the null (intercept only) model using the method described by Liang, Paulo, Molina, Clyde, and Berger (2008), as implemented in the BayesFactor package for R (Morey & Rouder, 2015). The Bayes Factors were used to calculate the posterior probability of each regression model and the null model via Bayes' Theorem, assuming equal prior probabilities. $R^2$ and adjusted $R^2$ were calculated for each model.

These analyses were first applied to the mean estimates for each fraction across all children. As shown in Table 1, the hybrid model had the largest Bayes Factor and posterior probability, and it explained more variance than the fraction magnitude model, $F(2, 40) = 111.6$, $p < .001$, the componential model, $F(1, 40) = 716.4$, $p < .001$, and the null model, $F(3, 40) = 900.0$, $p < .001$. The hybrid model's predictions for mean estimates of the large and small component fractions are shown in Figure 3 (dashed lines).

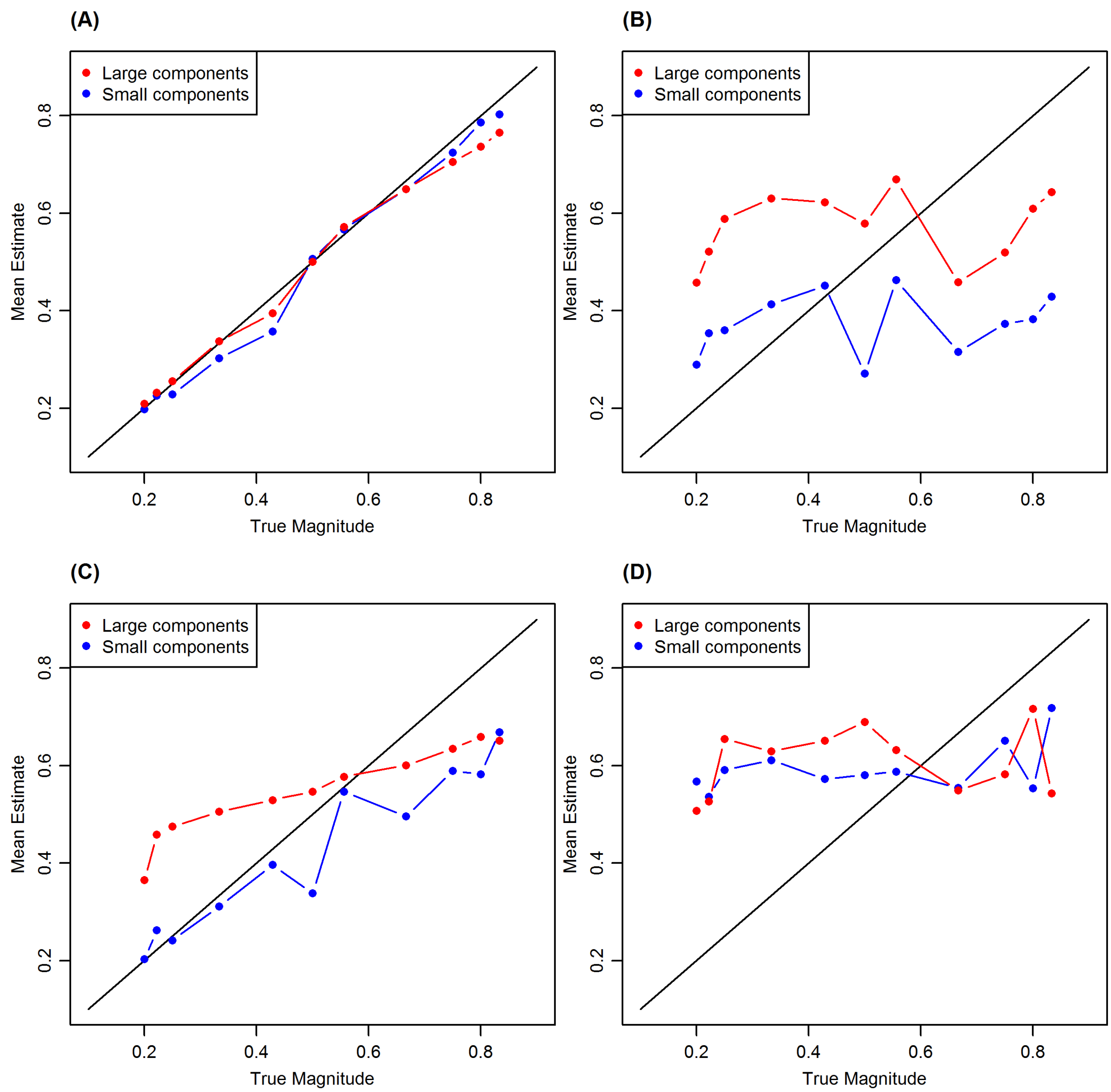
*Table 1.* Summary of linear regression fits. $\beta_{intercept}$, $\beta_{magnitude}$, $\beta_{numerator}$, and $\beta_{denominator}$ denote the coefficients of the corresponding predictors in each model. *BF* stands for Bayes Factor relative to the null model. *P*(M|D) stands for posterior probability of the model given the experimental data. Adj. $R^2$ stands for adjusted $R^2$.

| Model | $\beta_{intercept}$ | $\beta_{magnitude}$ | $\beta_{numerator}$ | $\beta_{denominator}$ | BF | *P*(M\|D) | $R^2$ | Adj. $R^2$ |
|---|---|---|---|---|---|---|---|---|
| Fraction magnitude | 0.182 | 0.616 | – | – | 3.6E19 | ~0% | 90.4% | 90.2% |
| Componential | 0.436 | – | 0.032 | -0.011 | 2.1E9 | ~0% | 72.4% | 71.0% |
| Hybrid | 0.067 | 0.702 | -0.001 | 0.004 | 1.5E33 | ~100% | 98.5% | 98.4% |
| Null | 0.492 | – | – | – | 1.0 | ~0% | – | – |

The same analyses were then repeated for each child. Average posterior probability across children was highest for the fraction magnitude model (54.8%), intermediate for the hybrid (22.2%) and componential (17.2%) models, and lowest for the null model (5.9%). Similar results were obtained when children were classified into groups by selecting the single most probable model for each child: most children were placed into the fraction magnitude group (54.0%), followed by the componential (21.2%), hybrid (16.8%), and null (8.0%) groups.

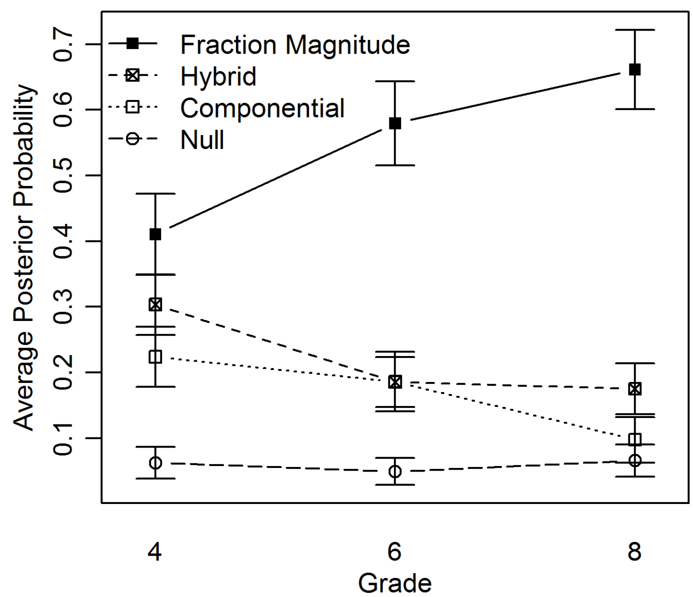
To illustrate the different response patterns characteristic of these four groups, Figure 5 shows mean estimates for large and small component fractions separately for children in each group. In sum, the fit of the hybrid model to the aggregated data was driven by a mix of distinct response patterns among individuals, rather than indicating consistent use of the hybrid model by all, or even most, children.

*Figure 5.* Mean estimates for large and small component fractions plotted against true magnitudes separately for children for whom the model with the highest posterior probability was (A) the fraction magnitude model, (B) the componential model, (C) the hybrid model, and (D) the null model (Experiment 2). 45° diagonal lines indicate the locations of normatively correct estimates.




To further test the hypothesis of a shift with age and experience away from emphasis on the components of fractions and toward emphasis on fraction magnitudes, children's posterior probabilities for each model were submitted to linear regression with grade as a predictor. These analyses found an increase with grade in the probability of the fraction magnitude model, $\beta = 0.063$, $F(1, 135) = 7.94$, $p = .006$, and decreases with grade in the probabilities of the hybrid model, β = -0.032, $F(1, 135) = 4.76$, $p = .031$, and the componential model, β = -0.031, $F(1, 135) = 4.14$, $p = .044$ (Figure 6). The analyses also indicated that the data of a substantial minority of fourth graders were already best fit by the fraction magnitude model, and the data of substantial numbers of eighth graders were not best fit by that model.

*Figure 6.* Posterior probabilities of the fraction magnitude, hybrid, componential, and null models averaged across individuals for each grade level (Experiment 2). Error bars indicate standard errors.




**Discussion**

In Experiment 1, whole number bias on the fraction magnitude comparison task decreased from fourth to fifth grade, and was no longer significant in fifth grade. This finding alone could create the impression that the major developmental changes in whole number bias occur in the relatively short period from fourth to fifth grade. However, the results of Experiment 2 indicated that whole number bias in fraction number line estimation decreases over the more protracted period from fourth to eighth grade. This result highlights the importance of using multiple measures to assess the development of whole number bias. More general implications of the developmental changes identified in Experiment 2 are deferred to the General Discussion.

Bayes Factors analyses of group data yielded results consistent with a hybrid model of magnitude representation. However, parallel analyses of the data of individuals indicated that only a minority of children exhibited hybrid response patterns. The aggregated data concealed substantial diversity among individuals, much as has been found in other areas of mathematical cognition (Siegler, 1987, 1989). These findings caution against drawing inferences about the form of children's mental representations based on aggregated data alone.

Finally, effects of component size on number line estimates remained even when fractions that were in lowest terms were removed from the data. This result allows us to exclude a possible explanation for these effects in Experiment 1 – that they merely reflect a difference between fractions that are or are not in lowest terms – and thus supports our interpretation that the effects reflect influences of whole number bias.

## General Discussion

The present findings hold implications regarding the development of fraction magnitude knowledge, fraction equivalence, and numerical knowledge more generally.

### Development of Fraction Magnitude Knowledge

The present findings demonstrate that whole number bias decreases with children's age and experience with fractions. Effects of fraction component size decreased, and effects of fraction magnitude increased, between fourth and eighth grade on the number line task in Experiment 2. Even in the brief period between fourth and fifth grade, whole number bias decreased on the magnitude comparison task in Experiment 1 (the task was not presented in Experiment 2). Thus, although whole number bias continues even in adulthood (Bonato et al., 2007; DeWolf, Grounds, Bassok, & Holyoak, 2014; Fazio et al., 2016; Meert et al., 2009, 2010b; Obersteiner et al., 2013), the size of this bias shows a clear decline over the period during which children study fractions in school (CCSSI, 2010).

Our findings dovetail with those of a previous study involving fraction magnitude comparison (Gabriel et al., 2013). That study used the size of the distance effect on fraction magnitude comparisons to assess children's reliance on fraction magnitudes during the task. Distance effects on response time (though not accuracy) increased from fifth to seventh grade. The present study found analogous developmental trends not only in magnitude comparison but also in number line estimation, a finding that demonstrated that the previous findings were not attributable to use of strategies for magnitude comparison that circumvented consideration of fraction magnitudes. Instead, both previous and present findings seem to reflect increasing reliance on fraction magnitudes, and concurrently decreasing whole number bias.

The Bayesian model comparisons conducted in Experiment 2 permitted a more precise description of these representational changes. According to this analysis, for some children, whole number bias reflects reliance on componential representations of fractions. For other children, whole number bias reflects reliance on hybrid representations, which are influenced by both component sizes and fraction magnitudes, as proposed by Meert and her colleagues (Meert et al., 2009, 2010a, 2010b). Reliance on both of these forms of representation became less common with age and experience, whereas reliance on fraction magnitude representations, which are influenced only by fraction magnitudes, became more common. However, even in eighth grade, the estimated probability of using fraction magnitude representations was only 66% (Figure 6). Thus, the transition toward greater use of fraction magnitudes may continue beyond eighth grade, a possibility that should be investigated in future studies.

What could cause the transition from componential and hybrid to fraction magnitude representations? One possible cause is formal classroom instruction about fraction magnitudes. However, such instruction takes place primarily in the third and fourth grades in the U.S. (CCSSI, 2010), whereas the present findings and previous ones (Gabriel et al., 2013) suggest that reduction in whole number bias continues well beyond this period.

A possible explanation is that understanding of fraction magnitudes improves as a result of experience with related topics encountered after explicit instruction in fraction magnitudes: decimals, percentages, ratios, rates, proportions, and rational number arithmetic. Studying these topics might improve children's understanding of rational number magnitudes in general and that of fraction magnitudes in particular. Illustratively, fraction arithmetic experience indicates that component size is a poor predictor of fraction magnitude. Children might initially believe, for example, that 1/2 is smaller than 2/6, but exposure to an addition problem such as 2/6+1/6 = 1/2 could lead them to recognize that 1/2 must be larger than 2/6 because the sum of positive numbers must be larger than the addends. Consistent with the possibility that learning fraction arithmetic improves understanding of fraction magnitudes, in a study of U.S. and Chinese sixth graders, effects of national origin (U.S. or China) on fraction magnitude understanding were fully mediated by fraction arithmetic ability (Bailey et al., 2015).

It is uncertain whether all children pass through a "phase" of componential and hybrid representations. About 40% of fourth graders in Experiment 2 already exhibited fraction magnitude representations. These children may have made the transition away from componential or hybrid representations in earlier grades, or some children may never have relied on such representations. It is also uncertain whether the appearance of componential and hybrid representations constitutes a natural development or whether it is contingent on specific aspects of fractions instruction. For example, instruction that leverages children's early understanding of ratios and proportions (Boyer & Levine, 2015; McCrink & Wynn, 2007; Mix, Levine, & Huttenlocher, 1999) might accelerate the transition to fraction magnitude representations, or even allow children to avoid componential and hybrid representations entirely. These also are important questions for future research.

**Development of Fraction Equivalence**

Whole number bias implies that equivalent fractions with different component sizes are not represented as equal. Consistent with this implication, children in the present study estimated that fractions with larger components were larger than equivalent fractions with smaller components. These results complement previous ones regarding children's difficulties with fraction equivalence (Bright et al., 1988; Byrnes & Wasik, 1991; Kamii & Clark, 1995; Kerslake, 1986; Ni, 2001). They also extend these findings by showing that equivalent fractions are not represented as equal even when presented separately, without the need for comparison or conversion between them.

However, it is concerning that the influence of whole number bias on representations of equivalent fractions persisted through sixth grade, and for about one quarter of children, into eighth grade. Children are expected to learn procedures that depend on fraction equivalence in fourth and fifth grade (CCSSI, 2010). The typical procedures for adding fractions with unlike denominators, reducing fractions to lowest terms, and canceling during fraction multiplication would make no sense if equivalent fractions were not equivalent. For example, if 3/6 were not equivalent to 1/2, how would it be legitimate to reduce 3/6 to 1/2? Poor understanding of fraction equivalence could impede understanding of related fraction procedures when they are taught, even if understanding of fraction equivalence improves later. Moreover, studies with adults, even expert mathematicians, indicate that the whole number bias never goes away entirely (Alibali & Sidney, 2015; Obersteiner et al., 2013). These observations highlight the importance of developing interventions specifically aimed at improving understanding of fraction equivalence before children are taught procedures that are built on fraction equivalence (e.g., Hunt, 2013).

Knowledge of whole numbers may interfere with understanding of fraction equivalence in another way as well. Whole numbers stand in one-to-one correspondence with their magnitudes, and thus do not have equivalents. To understand rational number equivalence, children must learn that this one-to-one relation is not true of fractions: multiple distinct fractions represent the same magnitude. Children often assume that properties of whole numbers are also true of fractions, and abandon such beliefs slowly if at all (Ni & Zhou, 2005). For example, children experience difficulty understanding that fractions are *dense* – that is, no matter how small the distance between two fractions, there are fractions in between – because this property violates the assumption, based on whole number experience, that every number has a unique successor (Smith et al., 2005; Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2004). Similarly, understanding of fraction equivalence may develop slowly in large part because it violates the assumption that distinct numbers must represent distinct magnitudes.

**Implications for Numerical Development**

The developmental trajectory of whole number magnitude representations has often been described in terms of a logarithmic-to-linear shift. Accurate (linear) representations of whole number magnitudes are preceded by logarithmic representations characterized by exaggerated distances between smaller numbers and compressed distances between larger numbers (Ashcraft & Moore, 2012; Berteletti, Lucangeli, Piazza, Dehaene, & Zorzi, 2010; Berteletti, Lucangeli, & Zorzi, 2012; Booth & Siegler, 2006; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Geary, Hoard, Nugent, & Byrd-Craven, 2008; Opfer, Thompson, & Kim, 2016; Siegler & Booth, 2004; Siegler & Opfer, 2003). By contrast, the present findings suggest that development of fraction magnitude understanding involves a shift from componential or hybrid representations towards fraction magnitude representations, a "componential-to-fraction magnitude shift."

Previous studies have found no evidence for logarithmic representations of fractions among children (Iuculano & Butterworth, 2011; Siegler et al., 2011). Further, logarithmic models of numerical magnitude representation cannot in principle capture the effects of component size observed in the present study, because these models posit that the represented magnitudes of numbers depend only on their true magnitudes, not on the sizes of their components. The same constraint applies to proportion estimation models, which have been proposed as an alternative to logarithmic models in research on whole number development (Barth & Paladino, 2011; Rouder & Geary, 2014). Thus, the componential or hybrid representations that serve as developmental precursors to accurate fraction magnitude representations are fundamentally different from those that serve as developmental precursors of accurate whole number magnitude representations.

The different starting points of the logarithmic-to-linear and componential-to-fraction magnitude shifts may reflect differences in the developmental origins of whole number and fraction knowledge. Logarithmic representations of symbolic whole numbers are believed to originate with innate or early-developing representations of non-symbolic quantities, such as dot arrays (Dehaene, 2011; Piazza, 2010). In principle, representations of non-symbolic ratios, such as ratios between dot arrays, could play a similar role in the development of fraction knowledge (Matthews & Chesney, 2015; Matthews, Lewis, & Hubbard, 2016). However, the present findings suggest a different developmental precursor to accurate representations of symbolic fractions: knowledge of symbolic whole numbers. Children's knowledge about the whole number components of fractions gives rise to componential and hybrid representations that, for many people, eventually give way to fraction magnitude representations.

The componential-to-fraction magnitude shift appears to occur more slowly than the logarithmic-to-linear shift. In a number line estimation task using whole numbers from 0 to 1,000, the proportion of children whose estimates were best fit by a linear model increased from 9% in second grade to 72% in sixth grade – an increase of 63% (Siegler & Opfer, 2003). Opfer and Siegler (2007) found that many children shifted from logarithmic to linear representations in response to feedback after only one feedback trial! In Experiment 2 of the present study, the posterior probability of the fraction magnitude model increased from 41% in fourth grade to 66%

in eighth grade – an increase of only 25% over four years (Figure 6). Thus, the componential-to-fraction magnitude shift occurs more slowly than the logarithmic-to-linear shift and may not occur at all for some people.

In summary, both whole number and fraction development involve progression towards accurate representations of numerical magnitudes, characterized by a linear relation between represented and true magnitudes. However, the starting points of the progressions, the speed at which they occur, and the obstacles that learners must overcome all differ for whole numbers and fractions. Thus, development of numerical magnitude representations provides a unifying theme within which to characterize both similarities and differences between fraction and whole number development.

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## Endnote

1. Both sets of fractions were required to include a pair of fractions with magnitudes equal to 0.5. However, only one set could include the fraction 1/2, because no fraction was allowed to appear in both sets. Thus, in the other set, 2/4 was treated as the small-component fraction in the pair equal to 0.5.
2. The magnitude comparison task included only comparisons in which the larger fraction's numerator and denominator were both smaller or both larger than those of the smaller fraction. We suspect that comparisons in which the larger fraction has a larger numerator but a smaller denominator (e.g., 3/4 vs. 2/5) would elicit results intermediate to those found for the two types of comparison included in the present study. However, to our knowledge, no study of fraction magnitude comparison among children has included all three of these types of comparison. For findings regarding comparisons between fractions with equal numerators or equal denominators, see Meert et al. (2010b).