Siegler, R., Fuchs, L., Jordan, N., Gersten, R., & Ochsendorf, R. (2015). The Center for Improving Learning of Fractions: A progress report. In S. Chinn (Ed.), *The Routledge international handbook of dyscalculia and mathematical learning difficulties*, (pp. 292-303). New York: Routledge.

The Center for Improving Learning of Fractions

A progress report

Robert Siegler, Lynn Fuchs, Nancy Jordan, Russell Gersten, and Rob Ochsendorf

This chapter has two interrelated purposes. The first, and primary, goal is to report findings from our Center regarding the development of understanding of fractions and the effectiveness of interventions based on those findings. The second goal is to describe how research centers that bring together investigators with complementary skills and knowledge can pursue objectives of scholarly and public importance that none of the investigators could reach individually.

History of the project

In the last ten years, knowledge on the cognitive underpinnings of learning difficulties in mathematics has increased substantially (e.g., Berch & Mazzocco, 2007; Geary, 2010). These advances have been achieved in large part through the application of cognitive theory and experimental methods to understanding cognitive processing in individuals with such difficulties. For example, sophisticated experimental paradigms have revealed that children with mathematics difficulties often have limited working memory resources relative to peers without such difficulties (e.g., Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Swanson & Jerman, 2006).

Despite these advances in our understanding of cognitive processes underlying mathematics difficulties, relatively little work has been done to capitalize on this research and develop innovative strategies for improving instruction for students who struggle to learn mathematics – in particular, for students with learning disabilities in mathematics or those who are at-risk for developing learning disabilities in mathematics.

To complement its existing research programs in mathematics education, in 2010 the National Center for Special Education Research (NCSER) at the Institute of Education Sciences (IES) held a competition for a research and development center on improving mathematics instruction for students with mathematics difficulties. The

Request for Applications stipulated that the center would conduct a focused program of research utilizing cognitive science to develop innovative approaches to improving instruction for students with learning difficulties in mathematics. The focused program of research was intended to extend scientific knowledge of the underlying cognitive processes that contribute to learning difficulties in mathematics in order to identify new approaches for intervening and providing more effective instruction for these students. IES has demonstrated a strong commitment to funding research projects that make use of cognitive science principles to help tackle the challenges inherent in formal education. It was the Institute's hope that such an interdisciplinary approach would lead to effective innovations in mathematics instruction for students with disabilities.

As a result of this competition, in 2010 IES announced a five-year, $10 million cooperative agreement with the University of Delaware to launch the Center for Improving Learning of Fractions. Fractions seemed a particularly important focus for this Center, because fractions are so difficult for so many children. For example, despite children in the U.S. receiving substantial fractions instruction *beginning in third or fourth grade* (NCTM, 2006), the 2004 National Assessment of Educational Progress (NAEP) revealed that 50% of *eighth graders* could not correctly order three fractions (2/7, 1/12, and 5/9) from least to greatest. The difficulty continues in high school and college; for example, on another NAEP item, fewer than 30% of 11th graders translated 0.029 into the correct fraction (Kloosterman, 2010). Community college students also struggle with fractions; a sample of such adults could identify the larger of two fractions on only 70% of trials, despite chance yielding 50% correct performance (Schneider & Siegler, 2010).

The Center's core activities involve three simultaneous strands of research: (1) small-scale experimental studies examining the cognitive processes underlying magnitude representations of fractions, (2) a longitudinal study to identify key predictors of learning difficulties with fractions, (3) development and testing of an intervention focused on improving understanding of fraction magnitudes. Different co-PI's on the grant had expertise in each of these three areas – Siegler in small experimental studies and the theoretical analyses they are designed to test, Jordan in large-scale longitudinal studies, and Fuchs in randomized controlled trial intervention studies. In addition, Russell Gersten has expertise in all of these areas, as well as in dissemination of research. The remainder of this chapter will focus on the considerable progress that we have made in each of these strands.

Theoretical analysis and small-scale experimental tests

Underlying the research in all three strands of our project is the integrated theory of numerical development (Siegler, Thompson, & Schneider, 2011). Within this theory, fractions are viewed as crucial not only because they express values that cannot be expressed with whole numbers but also because they provide most children's first opportunity to expand their understanding of numbers beyond the properties of whole numbers. All whole numbers can be represented by a single numeral, have unique successors, never decrease with multiplication, never increase with division, and so on. Many children naturally assume that these are properties of all numbers, because all of their numerical experience has been with whole numbers. However, none of these properties is true of fractions or of numbers in general. Instead, the only property that all real numbers have in common is that they have magnitudes that can be located and ordered on number lines. Thus, understanding fractions contributes to the process of numerical development both by indicating that many properties that are true of all whole numbers are not true of numbers in general and by indicating that one quality unites all types of real numbers – that they possess magnitudes and can be ordered on number lines.

Understanding fractions requires both conceptual and procedural knowledge. Conceptual knowledge of fractions involves knowing what fractions are: that they are numbers that stretch from negative to positive infinity; that between any two fractions are an infinite number of other fractions; that the numerator-denominator relation, rather than either number alone, determines fraction magnitudes; that fraction magnitudes increase with numerator size and decrease with denominator size; that fractions can be represented as points on number lines; and so on.

A particularly important type of conceptual understanding of fractions is knowledge of their magnitudes. Several measures of fraction magnitude knowledge, including number line estimation, magnitude comparison, and ordering of several fractions, correlate highly with proficiency at fraction arithmetic and overall mathematics achievement among people 10 years and older (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Jordan, et al., 2013; Mazzocco & Devlin, 2008; Siegler & Pyke, 2013; Siegler et al., 2011, 2012). The relation between fraction magnitude knowledge and overall mathematics achievement remains strongly present even when procedural knowledge (fraction arithmetic competence) is statistically controlled.

Acquiring conceptual knowledge of fractions poses several challenges. A major source of difficulty acquiring conceptual knowledge of fractions is that children's massive prior experience with integers leads to a whole-number bias, in which properties of positive integers are incorrectly generalized to fractions (Ni & Zhou, 2005). For example, based on their knowledge of whole numbers, even high school students often claim that there are no numbers between fractions such as 5/7 and 6/7 (as there are no integers between 5 and 6), and that multiplication cannot yield products smaller than both multiplicands (Vamvakoussi & Vosniadou, 2004, 2010). A related difficulty comes in understanding how the relation between numerator and denominator, rather than either alone, determines fraction magnitude – for example, that $2/3 > 5/9$, even though both the numerator and denominator of 2/3 are smaller (Fazio, Bailey, Thompson, & Siegler, 2014).

Procedural knowledge of fractions – knowledge of fraction arithmetic – is also difficult. One reason is that fraction arithmetic violates several patterns that children observe with whole numbers. Learning that multiplication can produce answers smaller than either multiplicand (e.g., $1/2 * 1/4 = 1/8$) is difficult to grasp for children who conclude from their experience or teacher statements that multiplication cannot have this effect. Another source of difficulty is that fraction arithmetic procedures overlap in complex ways that underlie several widespread types of errors. For example, maintaining the same denominator is appropriate when adding or subtracting fractions (e.g., 3/5 − 1/5 = 2/5), but it is inappropriate when multiplying or dividing them (e.g., 3/5 * 1/5 ≠ 3/5). Children often become confused regarding the operations that require equal denominators and what to do on problems that have equal denominators when the operation does not require them (as in 3/5 * 1/5). Similarly, performing the same operation on numerators and denominators independently leads to correct multiplication answers (e.g., 1/2 * 1/3 = (1*1)/(2 * 3) = 1/6), but operating independently on numerator and denominators leads to incorrect addition and subtraction answers (e.g., 1/2 + 1/3 ≠ 2/5). Conceptual understanding of why some fraction arithmetic procedures are appropriate and others inappropriate would almost certainly reduce or eliminate such errors, but few U.S. teachers have such understanding themselves (unlike East Asian teachers, who typically do possess this conceptual understanding of fraction arithmetic procedures; see Ma (1999) and Moseley, Okamoto & Ishida (2007)).

Consistent with the central role accorded to fractions in the integrated theory of numerical development, failure to master fractions has large consequences. It impedes acquisition of more advanced mathematics (National Mathematics Advisory Panel, 2008); indicative of this phenomenon, a nationally representative sample of 1,000 U.S. Algebra 1 teachers rated fractions as one of the two largest weaknesses in students' preparation for their course (NORC, 2008). Failure to master fractions also precludes participation in many remunerative and satisfying occupations (McCloskey, 2007).

Consistent with these informed opinions, a variety of studies have found that fraction knowledge is closely related to overall mathematics achievement, both when the two are measured at the same age and when earlier knowledge of fractions is related to later mathematics achievement (Bailey et al., 2012; Booth & Newton, 2012; Mazzocco & Devlin, 2008; Siegler & Pyke, 2013; Siegler et al., 2011). For example, analyses of two large, longitudinal data sets, one from the U.S. and one from the U.K., indicated that fifth graders' knowledge of fractions uniquely predicted those students' knowledge of algebra and overall mathematics achievement in high school 5–6 years later, even after statistically controlling for other mathematical knowledge, verbal and nonverbal IQ, reading comprehension, working memory, and family income and education (Siegler et al., 2012).

Children with difficulties learning mathematics in general tend to have particular problems learning fractions. One example of this phenomenon was evident in a study of sixth and eighth graders' knowledge of fractions (Siegler & Pyke, 2013). Typically achieving sixth graders (those in the top two-thirds of their age group on mathematics achievement test scores) were already more advanced in their knowledge of fraction arithmetic than low-achieving peers, and the typically achieving students' fraction arithmetic knowledge increased substantially between sixth and eighth grade, from 49% to 73% correct. In contrast, low-achieving children, those in the bottom one-third of their age group on the same achievement test, were already behind in sixth grade and only increased from 32% to 40% correct from sixth to eighth grade. This much slower rate of progress occurred despite all of the children being in the same classrooms and having the same teachers and textbooks. Thus, at least in the U.S., children generally have difficulty acquiring conceptual and procedural knowledge of fractions, and children with math learning difficulties have particular difficulty.

Longitudinal tests of theoretical predictions

The Delaware longitudinal study is examining the development of fraction concepts and procedures from third through sixth grade. The goal is to identify key component processes and skills that predict or underlie fraction learning. Tests were presented to 481 children in the winter of their third grade year (2010–2011). Drawn from two school districts in Delaware (U.S.A.), the children represented a diverse range of ethnicities, SES status, and ability levels. Because we were especially interested in children at-risk for learning difficulties or disabilities in mathematics, we oversampled in schools located in low-income communities (i.e., about 60% of children in our sample were identified as low-income based on their participation in free or reduced lunch programs).

Based on the integrated theory of numerical development, we hypothesized that accurate representations of numerical magnitudes would be uniquely important for acquisition of fraction knowledge. To test this premise, we examined the degree to which domain-general cognitive processes and number-specific knowledge in third grade predicted fraction outcomes at the end of fourth grade, after children had finished their first year of formal instruction on rational numbers (Jordan et al., 2013). General predictors included attentive behavior in the classroom, working memory, language, reading fluency, and nonverbal reasoning; number-specific predictors were number line estimation of whole numbers (0 to 1000), approximate number system acuity (e.g., rapid distinction between sets of items without counting), and calculation fluency. Fraction outcomes included measures of fraction concepts (e.g., magnitude comparisons and equivalence judgments) and fraction procedures (i.e., computation with fractions).

Confirming our main hypothesis, ability to estimate placement of whole numbers on a number line was the most important contributor to both aspects of fraction knowledge. For fraction concepts, the full set of predictors accounted for 56% of the variance, with number line estimation, calculation fluency, language, nonverbal reasoning, and attentive behavior each making significant unique contributions. For fraction procedures, the set of predictors accounted for only 30% of the variance in performance, with number line estimation, working memory, attentive behavior, and calculation fluency each making unique contributions.

Why is facility on a number line estimation task that involves whole numbers an important predictor of knowledge of fractions, over and above general calculation skill? Both whole numbers and fractions have magnitudes that can be located on number lines (Siegler et al., 2011). Insights with whole numbers give children an advantage in learning fraction concepts as well. Moreover, proportional reasoning, which underpins fraction understanding, may be involved in the whole-number line estimation task (Barth & Paladino, 2011).

In a follow-up study, one year later (end of fifth grade), we examined whether ability to estimate whole-number magnitudes on a number line continues to predict fraction outcomes, or whether other fraction-relevant processes supersede whole-number understanding (Hansen et al., under review). We re-administered the number line estimation task with whole numbers to the same group of children early in fifth grade, along with tasks assessing the ability to estimate fractions on a number line (ranging from 0 to 1, 0 to 2, and 0 to 5) and to judge proportional equivalence of visually depicted proportions (Boyer & Levine, 2012). Estimating on a fraction number line task requires more strategic analysis than on a whole-number line. To locate 2/3 on a 0–2 number line, for example, students must understand that 2/3 is less than one, in an absolute sense. They might divide the line in half to find the location of 1, then divide one into thirds, and then place their mark at 2/3. The proportional reasoning task draws on spatial processes specifically related to scale relations and multiplicative reasoning, which are important for understanding the concept of fraction equivalence (e.g., 1/3 is the same as 3/9; Boyer & Levine, 2012; Gunderson, Ramirez, Beilock, & Levine, 2012).

Multiple regression analyses revealed that whole-number magnitudes, fraction magnitudes, and proportional reasoning all made unique contributions to fraction concepts (once again controlling for general cognitive processes). However, only fraction magnitudes contributed independently to fraction arithmetic procedures. These findings suggest that whole-number magnitude knowledge contributes to acquisition of fraction magnitude knowledge, which in turn contributes to learning of fraction arithmetic.

The importance of both whole-number and fraction magnitude knowledge to fraction achievement has implications for intervention research and instructional practice. It supports a measurement approach to teaching rational numbers. Students who develop an understanding that all real numbers, including fractions, are assigned to their own location on the number line have an advantage in learning not only fractions but also algebraic concepts (Booth, Newton, & Twiss-Garrity, 2014; Siegler et al., 2011). Relations among proportional reasoning, multiplicative reasoning, and fraction equivalence also should be considered in intervention programs.

An intervention based on the theoretical analysis and empirical findings

The integrated theory of numerical development, the results of experimental studies, and the results of longitudinal studies converge on the conclusion that understanding of fraction magnitudes is crucial for both conceptual and procedural knowledge of fractions and for subsequent mathematics learning. Therefore, in a series of three randomized control trials, we have assessed the efficacy of an intervention, *Fraction Face Off*, which focuses heavily on inculcating understanding of fraction magnitudes. In each study, we contrasted the effects of this core program against a business-as-usual control condition that largely addressed the part-whole interpretation of fractions – the dominant representation of fractions in the U.S. mathematics curriculum.

In Study 1 (Fuchs, Schumacher, Long, et al., 2013), students were randomly assigned to two conditions, the core fraction intervention and a business-as-usual control condition. In Study 2, Fuchs, Schumacher, Sterba, et al. (2014) contrasted two variants of the core intervention program against the business-as-usual condition. One variant included activities aimed at building fluency with four measurement interpretation topics; the other variant included activities designed to consolidate understanding (rather than build fluency) with the same four topics. In the third study (Fuchs, Schumacher, Long, et al., 2013), we again randomly assigned students to three conditions: two variants of the core program and a business-as-usual control group. This time, one variant incorporated a word-problem intervention designed to encourage multiplicative thinking, while the other included a word-problem intervention requiring additive thinking.

In this brief overview, we focus on the overall effect of the core program versus the business-as-usual control group, as illustrated in the Year 1 study. (For additional information on the core *Fraction Face Off* program, and also for information on strategies to extend the core program using findings from Studies 2 and 3, consult the cited research reports.)

The three studies shared six key design features. First, students were at-risk fourth graders, with risk operationalized as performance on whole-number calculations below the 35th percentile at the start of fourth grade. To ensure the full distribution of risk status, we sampled students so that half of the sample in each study was below the 15th percentile, and the other half was between the 16th and 34th percentiles.

Second, we randomly assigned students to conditions, while stratifying by risk status and classroom. Third, interventions occurred in small groups, with three 30-minute sessions per week for 12 weeks. Fourth, we pre- and post-tested students on the measurement interpretation of fractions (using the fraction number line task, as described earlier in this chapter), calculation skill (adding and subtracting fractions), and released fraction items from the NAEP (easy, medium, and hard fourth-grade items as well as easy eighth-grade items). In Study 3, we also measured performance on fraction word problems that tapped multiplicative and arithmetic reasoning. Fifth, we evaluated the fidelity with which the intervention was conducted by audiotaping every session; randomly sampling tapes to comparably represent tutors, students, conditions, and sessions; and coding the accuracy with which key intervention components were implemented. In each study, fidelity was strong. Finally, we indexed the pre- and post-test performance of low-risk classmates (>34th percentile at the start of the study) to gain insight into how the intervention affected the achievement gap for at-risk versus low-risk students on fractions.

The major focus of the core fraction intervention program, as mentioned, is the measurement interpretation of fractions, which focuses primarily on representing, comparing, ordering, and placing fractions on number lines. This focus is supplemented by attention to the part-whole interpretation (e.g., showing objects with shaded regions and enumerating them relative to the total number of regions) and "fair shares" representations to build on prior knowledge and classroom instruction. Number lines, fraction tiles, and fraction circles are used to explain concepts throughout the 36 lessons, with a greater emphasis on these visuals at the start of the program. We start with proper fractions and fractions equal to one and midway through, we introduce improper fractions >1 and <2. We teach students to convert between improper fractions and mixed numbers, to place fractions on a 0–2 number line, and to order, compare, add, and subtract improper fractions and mixed numbers. We also focus on adding and subtracting fractions, but approximately 85% of content is allocated to understanding fractions rather than fraction arithmetic.

Each 30-minute lesson comprises four activities, with activity names reflecting the *Fraction Face Off* sports theme. In "Training," tutors introduce concepts, skills, problem-solving strategies, and procedures, relying on manipulatives and visual representations. "The Relay" involves group work on concepts and strategies taught during that day's Training. Students take turns completing problems while explaining their work to the group. All students simultaneously show work for each problem on their own papers. The third activity, "Sprint," provides supplementary activities designed to promote fluency with four key measurement topics. In the final activity, "The Individual Contest," students independently complete paper-pencil problems on content representing that day's and previous Training topics.

In each study, on each outcome, results favored students in the core fraction intervention program over those in the business-as-usual control group. For example, in Study 1, on comparing fraction magnitudes, the effect size (ES) favoring the intervention was 1.82 *SDs*; on the fraction number line task, it was 1.09. On comparing fractions, intervention students initially performed 0.12 *SDs* behind low-risk classmates, but completed the intervention 1.04 *SDs* ahead. By contrast, the achievement gap for control students increased from .05 to 0.42 *SDs*. (We did not collect fraction number line data on low-risk classmates, but post-test performance of intervention students was at the 75th percentile for a normative sample of sixth-grade students, as per Siegler & Pyke (2013).) NAEP effects were also significant and strong. The ES favoring intervention over control children was 0.92 *SDs*. The achievement gap between high- and low-risk children in the control condition remained large (1.09 at pre-test; 0.96 at post), while the gap for intervention students decreased substantially (from 1.07 to 0.08).

On calculations, effects again favored the intervention over the control condition. Here the ES was 2.50; the achievement gap between intervention students and low-risk classmates narrowed, while the gap for control students increased; and intervention students' post-test fraction arithmetic performance exceeded that of low-risk classmates. Given that classroom instruction allocated substantially more time to fraction arithmetic calculations than did the intervention, this suggests that understanding of the measurement interpretation transfers to procedural skill, at least for adding and subtracting fractions (e.g., Hecht, Close, & Santisi, 2003; Mazzocco & Devlin, 2008; Rittle-Johnson, Siegler, & Alibali, 2001; Siegler et al., 2011). This finding has practical significance and is supported by instructional theory (Siegler et al., 2011).

Importantly, analyses revealed that improvement in the measurement interpretation of fractions (but not improvement in the part-whole interpretation of fractions) mediated the effects of the intervention. This supports the hypothesis that the measurement interpretation is important to the development of students' fraction knowledge and suggests the need to reorient fraction instruction in the U.S. to include a dominant focus on the measurement interpretation of fractions.

Conclusions

Two hypotheses have guided the research described in this chapter. The first was a belief that knowledge of fractions is critical for success in more advanced mathematics, especially for success in algebra. The second was the critical importance of teaching students the measurement interpretation of fractions (i.e., that fractions can be represented as points on a number line). Results from the first several years of research have supported both these positions.

Knowledge of fractions is critical for success in algebra

The National Mathematics Advisory Panel (2008), on which two of the authors of this chapter served, reached this conclusion based on input from mathematicians, cognitive psychologists, mathematics education researchers, and mathematics teachers. The reasoning was that the level of abstraction required to truly understand fractions is essential before students can grasp the even more abstract notions of functions that is the core of algebra.

At that time, however, there was no empirical support for this assertion. However, since then, as noted earlier in this chapter, a host of longitudinal research has supported the critical role that knowledge of fractions at the end of elementary school and beginning of middle school plays in subsequent success in algebra and more advanced mathematics. Thus, the focus on improving the teaching of fractions and providing effective interventions for students who are in the "at-risk" category in the critical years for fractions instruction (grades 4, 5, and 6) seems well supported. The three intervention studies discussed in the previous section indicate that effective curricula can be developed for this population that focus heavily on building students' facility to understand and work with the measurement interpretation of fractions.

Importance of the measurement interpretation of fractions

The three strands of research at the Center all concentrate heavily on the importance of developing a linear representation of fraction magnitudes, that is, one in which a child's subjective representation of fractions' magnitudes increases linearly with the objective size of the fractions' magnitudes. This knowledge is often referred to as the *measurement interpretation* of fractions. Until recently, this emphasis was rare in American mathematics curricula in the elementary grades, but commonplace in many Asian curricula.

Case and colleagues (e.g. Okamoto & Case, 1996) hypothesized that development of increasingly sophisticated mental number lines was a key milestone in the development of number sense. The research described in this chapter strongly supports that insight. The longitudinal research conducted by both Siegler and Jordan and their colleagues demonstrates consistently strong predictive validity for measures involving placing fractions accurately on a number line, above and beyond general cognitive abilities and proficiency with computation.

A few issues come to mind from these lines of research. The first issue is whether there are other ways to measure understanding of fractions, above and beyond the measures of number line estimation and magnitude comparison used in this line of research. Our longitudinal research raises several other fascinating issues. The first is the important role that number line estimation (as assessed in third grade) plays in predicting fifth-grade performance in all aspects of fractions. Fluency and precision with the number line appears to be linked, as Jordan suggests, to the beginning of proportional reasoning. Proportional reasoning is, of course, integral to understanding what fractions mean, and how to interpret and use them accurately and precisely. Another interesting issue raised by Jordan and colleagues' research is that third graders' mathematics knowledge predicts fourth graders' understanding of fractions far better than it predicts their procedural competence (56% vs. 30% of explained variance). This leads one to wonder whether conventional instruction does a better job teaching students algorithms for addition and subtraction of fractions than in building the essential understanding of fractions necessary for future success in mathematics. The intervention research conducted by Fuchs and colleagues demonstrates that, at least for at-risk students, facility with the number line and accurate locations of fractions on a number line can be taught if instruction is well designed, very systematic, and very extensive in the amount of practice it provides in use of the number line. This issue is critical information to share as schools begin to implement what, to many, will be a much more challenging mode of mathematics instruction than teaching whole numbers.

The research of the Center demonstrates how a team of researchers can center a wide array of different type of research designs upon a central topic. In this case, a major focus– though hardly the entire focus– of the three strands of research has been the importance of the measurement interpretation of fractions (i.e. understanding of how to place fractions accurately on a number line). Mathematicians and mathematics educators deem competence in this area critical and thus it seemed an appropriate focal point for the research. Each strand approached the issue from a different lens. The small-scale experimental research indicated that understanding fraction magnitudes is a particularly important part of conceptual understanding of fractions, one that contributes substantially to both fraction arithmetic and to overall mathematics achievement. It also showed that children with mathematics difficulties progress remarkably slowly in acquiring fraction understanding. The longitudinal research indicated how number line estimation ability predicts future success in learning fractions above and beyond other measures of mathematics proficiency and general cognitive abilities, and that fraction knowledge more generally predicts algebra proficiency and overall mathematics achievement years later. Finally, the measurement interpretation served as a major instructional target in the series of three intervention studies (all RCTs) conducted in the intervention strand. The interventions unambiguously demonstrated that at-risk students could benefit from instruction that is explicit and systematic and targets this difficult but critical part of fraction knowledge.

Taken together, research conducted under the auspices of the Center has provided empirical support for significant shifts in the teaching of fractions, especially for students in the at-risk category. It also shows promise for improving screening and assessment in mathematics for students in the upper elementary grades. More generally, the strategy of funding centers that combine the efforts of researchers with different types of expertise to pursue specific, educationally significant goals seems likely to be fruitful in many different areas, not just this one.

References

Bailey, D. H., Hoard, M. K., Nugent, L., & Geary, D. C. (2012). Competence with fractions predicts gains in mathematics achievement. *Journal of Experimental Child Psychology, 113*(3), 447–455.

Barth, H. C., & Paladino, A. M. (2011). The development of numerical estimation: evidence against a representational shift. *Developmental Science, 14*(1), 125–135.

Berch, D., & Mazzocco, M. M. M. (Eds.). (2007). *Why is math hard for some children?: The nature and origins of mathematics learning difficulties and disabilities.* Baltimore, MD: Brookes Publishers.

Booth, J. L., & Newton, K. J. (2012). Fractions: Could they really be the gatekeeper's doorman? *Contemporary Educational Psychology, 37*(4), 247–253.

Booth, J. L., Newton, K. J., & Twiss-Garrity, L. K. (2014). The impact of fraction magnitude knowledge on algebra performance and learning. *Journal of Experimental Child Psychology, 118*, 110–118.

Boyer, T. W., & Levine, S. C. (2012). Child proportional scaling: Is 1/3=2/6=3/9=4/12? *Journal of Experimental Child Psychology, 111*(3), 516–533. doi:10.1016/j.jecp.2011.11.001

Fazio, L. K., Bailey, D. H., Thompson, C. A., & Siegler, R. S. (2014). Relations of different types of numerical magnitude representations to each other and to mathematics achievement. *Journal of Experimental Child Psychology, 123*, 53–72.

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Hamlett, C. L., Cirino, P. T., Changas, P. (2013). Improving at-risk learners' understanding of fractions. *Journal of Educational Psychology, 105*(3), 683–700.

Fuchs, L. S., Schumacher, R. F., Sterba, S. K., Long, J., Namkung, J., Malone, A., Changas, P. (2014). Does working memory moderate the effects of fraction intervention? An aptitude-treatment interaction. *Journal of Educational Psychology, 106*, 499–514.

Geary, D. C. (2010). Missouri longitudinal study of mathematical development and disability. *British Journal of Educational Psychology Monograph Series II*, 7, 31–49.

Geary, D. C., Hoard, M. K., Byrd-Craven, J., Nugent, L., & Numtee, C. (2007). Cognitive mechanisms underlying achievement deficits in children with mathematical learning disability. *Child Development, 78*(4), 1343–1359.

Gunderson, E. A., Ramirez, G., Beilock, S. L., & Levine, S. C. (2012). The relation between spatial skill and early number knowledge: The role of the linear number line. *Developmental Psychology, 48*(5), 1229–1241.

Hansen. N., Jordan, N. C., Fernandez, E., Siegler, R., Fuchs, L., Gersten, R., & Micklos, D. (under review). General and math-specific predictors of sixth-graders' knowledge of fractions.

Hecht, S. A., Close, L., & Santisi, M. (2003). Sources of individual differences in fraction skills. *Journal of Experimental Child Psychology, 86*(4), 277–302.

Jordan, N. C., Hansen, N., Fuchs, L. S., Siegler, R. S., Gersten, R., & Micklos, D. (2013). Developmental predictors of fraction concepts and procedures. *Journal of Experimental Child Psychology, 116*(1), 45–58.

Kloosterman, P. (2010). Mathematics skills of 17-year-old students in the United States: 1978–2004. *Journal for Research in Mathematics Education, 41*(1), 20–51.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States.* Mahwah, NJ: Erlbaum.

Mazzocco, M. M. M., & Devlin, K. T. (2008). Parts and "holes": Gaps in rational number sense among children with vs. without mathematical learning disabilities. *Developmental Science, 11*(5), 681–691.

McCloskey, M. (2007). Quantitative literacy and developmental dyscalculias. In D. B. Berch & M. M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities* (pp. 415–429). Baltimore, MD: Paul H. Brookes Publishing.

Moseley, B. J., Okamoto, Y., & Ishida, J. (2007). Comparing U.S. and Japanese elementary school teachers' facility for linking rational number representations. *International Journal of Science and Mathematics Education, 5*(1), 165–185.

National Council of Teachers of Mathematics (NCTM). (2006). Curriculum focal points for pre-kindergarten through grade 8 mathematics. Washington, DC: National Council of Teachers of Mathematics. Pdf available at http://www.nctm.org/focalpoints/down-loads.asp.

National Mathematics Advisory Panel (NMAP) (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel.* Washington, DC: U.S. Department of Education.

Ni, Y. J., & Zhou, Y. D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist, 40*(1), 27–52.

NORC (2008). *National Survey of Algebra Teachers,* OMB No. 1875–0243. Prepared for the National Mathematics Advisory Panel.

Okamoto, Y., & Case, R. (1996). Exploring the microstructure of children's central conceptual structures in the domain of number. *Monographs of the Society for Research in Child Development, 61*(1–2), 27–58.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93*(2), 346–362.

Schneider, M., & Siegler, R. S. (2010). Representations of the magnitudes of fractions. *Journal of Experimental Psychology: Human Perception and Performance, 36*(5), 1227–1238.

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science, 23*(7), 691–697.

Siegler, R. S., & Pyke, A. A. (2013). Developmental and individual differences in understanding fractions. *Developmental Psychology, 49*(10), 1994–2004.

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology, 62*(4), 273–296.

Swanson, H. L., & Jerman, O. (2006). Math disabilities: A selective meta-analysis of the literature. *Review of Educational Research, 76*(2), 249–274.

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction, 14*(5), 453–467.

Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction, 28*(2), 181–209.