DOI: 10.1002/cad.20372

WILEY

REVIEW

Understanding development requires assessing the relevant environment: Examples from mathematics learning

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Abstract

Although almost everyone agrees that the environment shapes children's learning, surprisingly few studies assess in detail the specific environments that shape children's learning of specific content. The present article briefly reviews examples of how such environmental assessments have improved understanding of child development in diverse areas, and examines in depth the contributions of analyses of one type of environment to one type of learning: how biased distributions of problems in mathematics textbooks influence children's learning of fraction arithmetic. We find extensive parallels between types of problems that are rarely presented in US textbooks and problems where children in the US encounter greater difficulty than might be expected from the apparent difficulty of the procedures involved. We also consider how some children master fraction arithmetic despite also learning the textbook distributions. Finally, we present findings from a recent intervention that indicates how children's fraction learning can be improved.

KEYWORDS
children's learning, decimals, fraction arithmetic, fractions, mathematics, number, textbooks

Despite general agreement that children's learning is shaped by the environment, surprisingly few studies provide detailed assessments of the everyday environments in which children learn specific content. The relative scarcity of assessments of these environments is understandable, because in many domains, the relevant input is difficult to assess or unknown. However, studies that have thoroughly examined the natural environments in which learning occurs suggest that greater effort in this direction would pay large dividends.

A well-known study of language development (Hart & Risley, 1995, 2003) illustrates the benefits that can be gained by assessing, in detail, inputs from the everyday environment. They had observers tape record family interactions in homes of varied socioeconomic status for an hour each month during the period when children in their sample were 9–36 months old.

The headline finding was that children from professional families heard more than three times as many words spoken to them as children from families on welfare; moreover, extrapolating the observed differences from birth to age 18 produced a language gap of millions of words. Further detailed analyses revealed especially large gaps in the use of unusual words and complex syntax. Statements directing children to do or not do something, which Hart and Risley termed "business talk," were similarly abundant throughout numerous homes, but frequency of more elaborative statements, especially ones responding to what the child just said or commenting on what the child just did, differed greatly. Differences in linguistic input at home predicted differences in children's vocabularies and other important outcomes. Number of words spoken in the household before age 3 correlated roughly $r = .6$ with children's vocabulary at age 3, individual differences in vocabulary at age 3 strongly predicted differences in vocabulary at age 9, frequency of exposure to low-frequency words proved especially predictive of children's vocabulary at age 9, and amount of non-business talk in the home before age 3 correlated substantially with children's performance on an IQ test at age 9 (Hart & Risley, 1995, 2003; Snow & Beals, 2006).

Language input also influences learning of specific content. Pruden, Levine, and Huttenlocher (2011) found a roughly 100-fold difference between the highest and lowest number of spatial words (e.g., circle, little, side) spoken by parents to their children at ages 14–46 months. A mediation analysis indicated that children who heard more spatial words from parents used more spatial words themselves. In turn, the more spatial words that children produced, the better their performance was on a spatial transformation task at 54 months of age.

The relation between language input and content knowledge is causal as well as correlational. Guiding randomly chosen parent–child dyads to talk about spatial features in a museum boosted parent–child spatial talk and subsequently improved children's assembling of puzzles (Polinsky, Perez, Grehl, & McCrink, 2017). Similarly, prompting some randomly selected parents to talk about budgeting in a museum's mock grocery store increased parent–child number talk and children's attention to numbers relative to prompting other parents to talk about healthy eating at the grocery store (Braham, Libertus, & McCrink, 2018).

Language input is not the only area in which the benefits of detailed examination of learning environments have been demonstrated; analyses of infants' visual environments have yielded similar benefits. Through placing lightweight head-mounted cameras on infants' foreheads, investigators have been able to learn which objects attract infants' visual attention. These studies have revealed a variety of non-intuitive findings, including that infants look at obstacles in their paths more often when crawling than when walking (Kretch, Franchak, & Adolph, 2014); that in the first two years of life, visual input shifts from a focus on faces to a focus on hands (Fausey, Jayaraman, & Smith, 2016); and that infants attend three times as long when people act as causal agents as when they engage in self-propelled motion without any obvious causal purpose (Cicchino, Aslin, & Rakison, 2011). None of these findings was self-evident; for example, beginner walkers might have been expected to look at obstacles more often than crawlers do, because stepping on obstacles while walking produces harder falls. Similarly, infants might have been expected to attend more to self-propelled motion than to causal actions involving small movements, because the self-propelled motion is more visually apparent. These and other findings regarding infants' visual environments raise a host of important theoretical issues, such as the issue of what drives the changes in which objects attract infants' gazes.

Focusing on the specifics of the input that children receive also has proved to be useful for understanding kindergartners' math learning. In one remarkable study using data from the Early Childhood Longitudinal Study, Kindergarten (ECLS-K) cohort of 1998–1999, examination of math knowledge and classroom instruction at the beginning and end of kindergarten indicated that most classroom instruction was devoted to counting from 1 to 10 and identifying simple geometric shapes, *which more than 90% of children had mastered before they started kindergarten* (Engel, Claessens, & Finch, 2013). Very little time was devoted to topics in which most kindergartners lacked proficiency, in particular single-digit addition and subtraction. Time spent on counting and shape identification *negatively* predicted math learning during kindergarten, whereas time spent on addition and subtraction positively predicted learning. These findings were robust after controlling for variables such as the child's race, gender, and linguistic background; mother's income and education; total amount of time spent by the teacher on mathematics; and school variables such as percent of students eligible for free or reduced-price lunches. The findings have been replicated with a 2010–2011 sample (Engel, Claessens, Watts, & Farkas, 2016), showing little change despite extensive mathematics reform efforts in the period between the two samples.

Even seemingly trivial aspects of the learning environment can have surprising effects. For instance, the internal spacing on a page or screen of arithmetic problems involving both addition and multiplication (e.g., $3 + 4 \times 5 =$ ___) influences the speed and accuracy of performance on such equations (Landy & Goldstone, 2007a, b, 2010). Narrower spacing between the operation symbol and the surrounding operands increases the probability of performing that operation first. In the problem above, narrower spacing between "3+4" than between "4 × 5" (e.g., 3+4 × 5) would increase the likelihood of answering "35" rather than "23." Mathematics involves abstraction over irrelevant details, but this does not mean that students abstract over irrelevant details.

The goal of this article is to demonstrate in greater depth the insights that can be gained about development through detailed analyses of relevant environments. In particular, we examine insights that can be gained through detailed assessment of one aspect of the environment—textbook problems—into one aspect of development—acquisition of fraction arithmetic. We also report recent research on how detailed analyses of natural environments can inspire interventions that improve learning in this area.

1 | WHY FOCUS ON FRACTION ARITHMETIC?

Several considerations led us to focus on this area. Even beyond the general importance of mathematics proficiency for long-term academic and occupational success (Ritchie & Bates, 2013), fractions appear to be particularly important. Fractions knowledge is foundational for acquiring more advanced mathematics in school; knowledge of fractions in 5th grade uniquely predicts overall mathematics achievement in 10th grade, even after statistically controlling for numerous other demographic and knowledge variables, including proficiency with all four whole number operations (Siegler et al., 2012). The predictive relation between fifth graders' performance on fractions problems, most of which involved fraction arithmetic, and overall mathematics achievement 5 years later was evident in both the United States and the United Kingdom. The importance of fractions extends beyond mathematics: Fraction and decimal arithmetic were involved in more than half of the equations on the reference sheets for recent Advanced Placement tests in both physics and chemistry (College Board, 2014, 2015). Further, knowledge of fractions is essential in many workplaces: a national survey of more than 2,000 US workers (Handel, 2016) indicated that two thirds of employees used fractions in their work. Occupations in which employees report using fractions include not only ones that we usually think of as involving mathematics, such as engineering and science positions, but also a wide range of other occupations, including nursing, carpentry, auto mechanics, and modern factory work (Hoyles, Noss, & Pozzi, 2001; Sformo, 2008).

Another reason to focus on fraction arithmetic is that children have great difficulty learning it. When thousands of US eighth graders were asked on the 1978 National Assessment of Educational Progress (NAEP) whether 12/13 + 7/8 was closest to 1, 2, 19 or 21, only 24% answered "2" (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980); the most common answers were "19" and "21." This and similar findings triggered a variety of reform efforts to improve mathematics education, culminating in the Common Core State Standards. However, when Lortie-Forgues, Tian, and Siegler (2015) presented the same problem to eighth graders in 2014, more than three decades later, accuracy had increased only from 24% to 27% correct.

Such errors might be interpreted as implying that children do not learn correct fraction arithmetic procedures, but that interpretation is only partially correct. The same children who use flawed strategies and generate implausible errors on some trials use correct strategies and answer correctly on other trials. In Siegler and Pyke (2013), for example, most sixth and eighth graders who were presented pairs of virtually identical fraction arithmetic problems (e.g., $3/5 \times 1/5$ and $3/5 \times 4/5$) used different strategies on at least one pair of the highly similar problems; 65% of such differing pairs of strategies included both a correct strategy and an incorrect one. Equally striking, children were only slightly more confident in their correct than in their incorrect answers. Together, these findings suggest that children learn both correct and incorrect strategies but are unable to identify through reasoning which are correct, leading to a competitive retrieval process without a filter for rejecting incorrect strategies when they are retrieved.

Why is learning about fractions and fraction arithmetic so difficult? At a general level, there are two types of reasons: inherent and culturally contingent (Lortie-Forgues et al., 2015). Inherent sources of difficulty are ones that would make learning difficult in any society in any time period. Among the inherent reasons for difficulty in learning fraction arithmetic are the difficulty of estimating the magnitudes of individual fractions, the complex relations between rational and whole number arithmetic operations, and the complex relations among different rational number arithmetic operations. An instance of the challenge of understanding fraction magnitudes is that evaluating the plausibility of "1/2 + 1/2 = 2/4" requires knowing the magnitudes of "1/2" and "2/4." An instance of the complex relations between arithmetic operations involving whole numbers and arithmetic operations involving fractions is that adding whole numbers involves performing the operation on all numbers in the problem (e.g., 3 + 5 + 4 + 5 = 17"), whereas adding fractions involves performing the whole number operation only on the numerators (3/5 + 4/5 = 7/5). An instance of the complex relations among different fraction operations is that children often overgeneralize from fraction addition to fraction multiplication, for instance incorrectly inferring that because 3/5 + 4/5 = 7/5, therefore $3/5 \times 4/5 = 12/5$. These inherent sources of difficulty would increase the difficulty of learning fraction arithmetic regardless of the era, society, or instructional approach presented to children.

Other sources of difficulty in fraction arithmetic are culturally contingent. These include teacher knowledge of the mathematical content, student knowledge of less advanced mathematics, societal pedagogical beliefs and values, and textbook presentations of the material. All of these vary among societies, subgroups within societies, and historical periods, and all influence learning.

2 | WHY FOCUS ON TEXTBOOKS?

The mathematics learning environment is multifaceted. It involves people who provide relevant input to children, such as teachers, parents, siblings, and classmates; sociological forces, such as racial, ethnic, social class, and gender biases within the child's society; educational policies at national, state, and local levels; and impersonal forces, such as technology, instructional materials, and games. Although all of these influences are important, this article focuses on a single aspect of the mathematics learning environment: distributions of problems in textbooks.

Textbooks offer several advantages for studying the impact of specific features of the environment on specific aspects of children's learning. The most popular textbook series in large countries such as China, India, and the United States are used by millions of children each year; thus, textbooks are an ecologically valid part of the learning environment. Many other types of materials are also used, including online resources, exercises prepared by one's colleagues, workbooks, and problems generated by teachers, parents, and tutors. However, the use of math textbooks is widespread in many countries around the world (Fan, Zhu, & Miao, 2013; Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002).

Another advantage of examining textbooks is that they indicate the order in which children encounter problems. Problems in an earlier grade textbook almost always are presented before ones in a later grade textbook, and teachers usually present chapters in the order in which they occur in the textbook (Freeman & Porter, 1989). A third advantage of studying textbooks as part of the learning environment is that the raw data are widely available; thus, replicating analyses of textbook problems and testing alternative interpretations by performing new analyses is easy. A fourth advantage is that examining textbooks allows investigation of a variety of educationally relevant issues (Fan et al., 2013): textbook features (e.g., analyses of the types of problems receiving more and less coverage); textbook use (e.g., analysis of how much teachers rely on textbooks), textbook goals (e.g., analyses of the most appropriate use of textbooks); and other issues regarding textbooks (e.g., how well different textbooks promote student learning). A fifth advantage is that parallel analyses of textbooks can be done cross-nationally. Textbooks are used throughout the world—a survey of fourth and eighth graders in more than 20 countries showed that 75% of students reported that their teachers primarily used textbooks for mathematics instruction (Horsley & Sikorová, 2014). This facilitates cross-national comparisons and allows tests of potential explanations of developmental differences in different societies.

Although we believe that problem distributions in textbooks exert a large influence on children's learning, it is important to note that this influence is depends on how teachers use the textbooks. Teachers select which problems to assign, and they decide how to teach the concepts and procedures relevant to the problems; this is why textbooks are sometimes viewed as mediators between the national curriculum, as defined by

TABLE 1 Percent of US textbook problems classified by arithmetic operation and denominator equality

	Arithmetic operation			
Denominator equality	**Addition**	**Subtraction**	**Multiplication**	**Division**
Equal denominators	12	13	1	1
Unequal denominators	13	12	29	19

curriculum standards, and the implemented curriculum, as enacted by teachers (Valverde et al., 2002). Modes of teaching are likely to interact with textbook characteristics. In particular, changing the distribution of problems in textbooks is likely to be most effective if teachers (and textbooks) call attention to important dimensions of variation among problems and explain clearly why those dimensions do or do not influence appropriate problem-solving procedures and ways of conceptualizing problems. This, in turn, implies that teacher education and professional development will also be important determinants of the gains that can be realized from changing the distribution of textbook problems.

3 | RELATION OF TEXTBOOK PROBLEM DISTRIBUTIONS TO CHILDREN'S PERFORMANCE

3.1 | Fraction arithmetic

3.1.1 | Textbook problems

To assess the distribution of fraction arithmetic problems in textbooks, Braithwaite, Pyke, and Siegler (2017) coded all symbolic rational number arithmetic problems presented in the fourth through sixth grade volumes of three popular US mathematics textbook series: Houghton Mifflin Harcourt's *GO MATH!* (Dixon, Adams, Larson, & Leiva, 2012a, b), McGraw Hill Education's *Everyday Mathematics* (University of Chicago School Mathematics Project, 2015a, b, c), and Pearson Education's *enVisionMATH* (Charles et al., 2012). These three series were chosen because all were among the most widely adopted US textbook series (Opfer, Kaufman, Pane, & Thompson, 2018) and had volumes for fourth through sixth grade, the grades in which fraction arithmetic receives the greatest emphasis in the United States (CCSSI, 2010). The problems were the full set of those that (a) had two operands, at least one of which was a fraction or mixed number, (b) were in symbolic form (i.e., with only numbers and operation specified), and (c) required exact answers (i.e., not estimates). Problems that had these characteristics constituted most of the problems in all textbooks that Braithwaite et al. (2017) analyzed, as well as in three other textbook series analyzed by Cady, Hodges, and Collins (2015).

The analyses of all three textbook series revealed strong associations between arithmetic operations and the types of operands (numbers) in the problems (Table 1). Among problems involving two fractions, only 4% of multiplication and division items but 50% of addition and subtraction items had equal denominators. The pattern was highly similar in all three textbook series.

Similar operand–operation associations were present in the distribution of problems having a fraction and a whole number (Table 2). Among problems with at least one fraction operand, only 5% of addition and subtraction items included a whole number operand (e.g., $6 - 3/5$), but 65% of multiplication and division items did (e.g., $6 \times 3/5$).

TABLE 2 Percent of US textbook problems classified by arithmetic operation and operand combination

	Arithmetic operation			
Operand number type	**Addition**	**Subtraction**	**Multiplication**	**Division**
Fraction–fraction	25	23	13	8
Whole–fraction	0	2	17	13

Note. Percentages may not sum to 100% because of rounding.

The associations between arithmetic operations and operands (numbers) in these problem distributions do not have any obvious mathematical justification. Children must learn to multiply fractions with identical denominators, just as they need to learn to multiply fractions with unequal denominators. Children need to learn to add and subtract problems with both whole numbers and fractions, just as they need to learn to multiply and divide them. The reason for the biases in textbook problems are unknown, but there is no question that the distribution of problems is far from random.

3.1.2 | Problems used in instruction

The fact that problems appear in textbooks does not guarantee that children will encounter them. Teachers do not typically present all problems in textbooks; they might compensate for the paucity of certain types of problems in textbooks by emphasizing them in class or homework assignments. This hypothesis was suggested to the first author by William McCallum, who took the lead role in writing the section of the Common Core State Standards on fractions instruction.

To test the assumption that textbook problems reflect the input children receive, Tian et al. (2019) asked 14 fourth, fifth, and sixth grade math teachers from five school districts in the greater Pittsburgh area to provide all problems that they presented to students in math class or as homework during the 2017–2018 school year. These problems were coded using the same criteria as in Braithwaite et al. (2017).

One main finding was that 70% of the fraction arithmetic problems in the in-class and homework assignments came from textbooks. Another main finding was that the fraction arithmetic problems that teachers assigned showed very similar distributions to those in the textbooks in Tables 1 and 1B. This was true for both the problems from textbooks that teachers assigned and the problems from sources other than textbooks. These findings supported our assumption that textbook problems are a good proxy for the problems that children encounter.

3.1.3 | Do children learn characteristics of problem input?

The fact that operations and operands are associated in textbook problems does not guarantee that children learn those associations. There are reasons to hypothesize that children would learn the associations between operations and operands that were present in the fraction arithmetic textbook items, but there are also reasons to hypothesize the opposite. On the one hand, learning of statistical associations between features of the environment is a fundamental learning mechanism that operates from infancy onward (Rakison, Lupyan, Oakes, & Walker-Andrews, 2008). There is no obvious reason to believe that this mechanism “turns off” in formal educational contexts. On the other hand, mathematics instruction emphasizes general principles and procedures, not associations between operations and operands, and textbooks and teachers focus learners' attention on those principles and procedures. This directing of attention might lead children to not detect mathematically irrelevant associations.

To determine which hypothesis was correct, Braithwaite and Siegler (2018) presented middle school students with two complementary types of problems. On generate-operand problems, an arithmetic operation was specified and children were asked to choose two numbers to accompany it (e.g., □ × □). On choose-operation problems, operands were specified and children were asked to choose an arithmetic operation to accompany them (e.g., 3/5□2/5).

Children's choices indicated that they had learned the operator–operand associations that were present in the textbooks. On the generate-operands task, when the specified operation was addition or subtraction, children usually generated pairs of fractions with equal denominators. When the specified operation was multiplication or division, they usually generated operand pairs with a whole number and a fraction. The choose-operation task yielded similar findings. When presented two fractions with equal denominators, children chose addition or subtraction more often than multiplication or division. In contrast, when presented a whole number and a fraction, they chose multiplication or division more often than addition or subtraction.

Children even learned the particular fractions that were most likely to appear in the textbooks. The frequency with which specific fractions were presented in textbooks correlated quite strongly ($r = .78$) with the frequency with which children generated them on the generate-operand problems. Part of this relation was due to fractions with small denominators appearing most often in both textbooks and children's choices, but even when denominator size was statistically controlled, the relation remained highly significant. Thus, children are exceptionally good at learning mathematically irrelevant characteristics of instructional input, such as relations between operations and operands and frequencies of particular fractions. Unfortunately, they are much less apt at learning desired procedures and concepts.

3.1.4 | Relations to children's fraction arithmetic performance

Frequencies of fraction arithmetic problems in math textbooks were highly predictive of middle school children's fraction arithmetic performance (Braithwaite et al., 2017). Most strikingly, despite correct multiplication and division procedures being identical for problems with equal and unequal denominators, children were considerably less accurate on the infrequently presented problems with equal denominators than on the frequently presented problems with unequal denominators. Thus, children in Siegler and Pyke (2013) were correct on 58% of multiplication items with unequal denominators (e.g., 3/5 × 1/4) but only 36% of multiplication items with equal denominators (e.g., 3/5 × 1/5). Other relations between frequency of different types of textbook problems and children's performance were also present. For instance, textbooks included far fewer division items than addition, subtraction, or multiplication items, and performance was far worse on division items than on problems involving any of the other three operations.

The textbook problem distributions were useful for explaining not only accuracy patterns but also common types of errors. For instance, the correct addition/subtraction rule (perform the operation on the numerators and maintain the denominator, as on 3/5 + 4/5 = 7/5) was frequently overgeneralized to multiplication (3/5 × 4/5 = 12/5). Such errors were as common as correct answers on equal-denominator multiplication problems in Siegler and Pyke (2013). These overgeneralization errors may have reflected the pattern of equal denominators appearing 10 times as often on addition/subtraction as on multiplication/division problems in textbooks. This statistical relation may have led children to infer that when denominators are equal, they should apply a version of the addition/subtraction rule to multiplication.

3.1.5 | Are these relations causal?

If the parallels between textbook problem distributions and children's performance reflect causal relations between them, then including more problems of the types that children rarely encounter at present could improve performance on these problems. Although experiments have not yet been conducted in which children are randomly assigned to receiving varying distributions of rational number arithmetic problems, findings from other areas of mathematics, such as mathematical equality, are encouraging.

Mathematical equality is the concept that the values on each side of the equal sign are equivalent. Despite this concept being fundamental to many areas of mathematics, and children frequently encountering the equal sign from kindergarten or first grade onward, understanding of it remains limited even years later. Most fourth graders, for example, answer problems such as "8 + 4 = □ + 5" by summing the numbers to the left of the equal sign (yielding the answer "12") or by adding all numbers in the problem (yielding the answer "17") (Falkner, Levi, & Carpenter, 1999). Both of these approaches indicate that children view the equal sign as a kind of "go" signal to execute the operation in the problem, rather than as a symbol indicating that the values on each side should be made equal.

Examination of elementary and middle school math textbooks yielded a hypothesis regarding the source of this delayed understanding. The textbooks rarely presented problems with operations on both sides of the equal sign. Only 5% of problems in middle school textbook series examined by McNeil et al. (2006) had operations on both sides of the equation (e.g., 4 + 5 = 2 + _). Moreover, such problems are comparably rare in elementary school textbooks (Powell, 2012). This finding suggested the hypothesis that lack of experience with such problems leads to children misinterpreting the equal sign.

To test whether a causal relation exists between encountering such non-standard problems and understanding the equal sign, McNeil, Fyfe, and Dunwiddie (2015) created a modified workbook with non-standard uses of the equal sign that are rarely found in standard workbooks. The modified workbook had the same number of problems as the standard one, but included several types of items that were absent from it. One type of problem included in the modified, but not the standard, workbook was items with operations on the right side of the equal sign (e.g., __ = 5 + 4). Another atypical type of item in the modified workbook replaced the equal sign with the words "is the same amount as." On both immediate and delayed posttests, children who had used the modified workbook showed better understanding of mathematical equivalence than peers who had used the standard workbook. These findings demonstrated a causal relation between encountering varied types of problems in workbooks and children's learning.

4 | DO BIASED PROBLEM DISTRIBUTIONS DOOM CHILDREN'S LEARNING?

The detailed parallels between textbook problem distributions and children's performance suggest that imbalanced distributions adversely affect children's learning of fraction and decimal arithmetic. However, some children do learn rational number arithmetic, despite encountering the same textbook distributions as their classmates. This raises the question of how they are able to do so.

An answer is suggested by data from China. Analyses of Chinese textbooks demonstrated similar problem distributions to those in US textbooks, and analyses of Chinese children's learning of the relations in textbooks between operands and operations showed that the Chinese children learned the relations at least as well as US children (Braithwaite & Siegler, 2018). In contrast to children in the United States, however, Chinese children solved fraction arithmetic problems very accurately. When presented with the same problems as children in Siegler and Pyke (2013), Chinese sixth graders answered 90% of items correctly (vs. 32% for US peers), and Chinese eighth graders answered 93% of items correctly (vs. 60% for US peers) (Bailey et al., 2015). The differences were even greater for children in the bottom one third of each distribution: 76% versus 15% correct for sixth graders, and 78% versus 20% correct for eighth graders. Thus, it seems as though being exposed to imbalanced fraction arithmetic textbook problem distributions does not doom children to poor fraction arithmetic performance.

This combination of findings raised two further questions: When do biased textbook problem distributions negatively influence children's learning? How do some children largely or completely avoid these negative effects, despite learning the associations that are present in the problem distributions?

5 | THE ROLE OF CONCEPTUAL UNDERSTANDING

One major determinant of when textbook problem distributions influence fraction arithmetic (and other areas) seems to be conceptual understanding of problem-solving procedures in the domain. In arithmetic, this type of conceptual knowledge concerns issues such as, "why does addition of positive numbers yield answers greater than either addend?" and "why does multiplication of numbers greater than one yield answers greater than either multiplicand but multiplication of numbers between 0 and 1 yield answers less than either multiplicand?" Contrasting rational number and whole number arithmetic illustrates the impact of conceptual understanding on children's learning.

We have already seen several illustrations of US students not applying conceptual knowledge to rational number arithmetic problems. When children claim that $12/13 + 7/8 \approx 19$ or that $6 + .32 = .38$, they are not considering whether the magnitudes of the answers to the problems make sense. When they claim that $1/2 + 1/2 = 2/4$, they are ignoring the principles that adding two positive numbers must result in an answer greater than either addend. When they claim that $3/5 \times 4/5 = 12/5$, they are ignoring (or do not know) the principle that multiplying two numbers between 0 and 1 must result in an answer less than either multiplicand.

A particularly dramatic illustration of lack of use of conceptual knowledge in fraction arithmetic comes from Braithwaite, Tian, and Siegler (2018). Sixth and seventh graders from US schools in a middle-income area were asked to estimate the magnitudes of individual fractions and sums of fractions on 0–1 number lines, and to estimate the magnitudes of whole numbers and sums of whole numbers on 0–1,000 number lines. Magnitudes of individual numbers were estimated first, followed by estimates of sums of pairs of numbers that had been presented previously. Thus, a child may have been asked to estimate the locations of 3/7 and of 4/9 earlier and of $3/7 + 4/9$ later. Fraction problems were converted into almost equivalent whole number problems by multiplying the nearest decimal equivalent of each fraction by 1,000 (e.g., for fractions 3/7, 4/9, and $3/7 + 4/9$, the parallel whole numbers were 429, 444, and 873). Answers to both fraction and whole number problems appeared equally often in each quartile of the number line.

Strikingly, on 52% of the fraction triads (a, b, and a + b), children estimated that one or both of the individual fractions were larger than the estimated sum of both fractions. Using the example above, this meant that roughly half of children estimated either 3/7 or 4/9 as being larger than 3/7 + 4/9. This pattern was observed on only 15% of whole number triads, demonstrating that the difficulty was not with children's understanding of the number line task or inattention to the task. The pattern was highly consistent across different triads of numbers: inaccuracy of estimates (percent absolute error) for each of the 16 fraction sums was greater than for all whole number sums and for 15 of the 16 individual fractions. Clearly, these children did not possess conceptual understanding of fraction addition or, at minimum, did not apply whatever conceptual understanding they had to estimating fraction sums.

Previous findings have indicated that children of similar ages also do not apply conceptual knowledge to evaluating the plausibility of answers to fraction arithmetic problems that require exact answers (Hecht, 1998; Hecht, Close, & Santisi, 2003; Siegler & Pyke, 2013). Moreover, addition is likely the easiest arithmetic operation to understand; understanding of subtraction, multiplication, and division is unlikely to be better. This lack of conceptual understanding opens the door to mathematically irrelevant associations based on textbook problem distributions influencing children's performance.

Chinese children appear to have greater conceptual, as well as procedural, knowledge of fractions. Although most US sixth and eighth graders incorrectly believe that the product of two positive fractions below one is larger than the larger operand alone (Siegler & Lortie-Forgues, 2015), most Chinese sixth and eighth graders do not commit this error (Tian & Siegler, 2018). Further, a far higher percentage of primary school mathematics teachers in China than in the United States can explain the rationales for fraction arithmetic procedures (Ma, 1999), making it possible for them to teach the rationales to their students. Chinese children's superior understanding of fraction arithmetic may allow them to override their associations between operations and operand features, thereby avoiding the detrimental effects of the spurious associations.

5.1 | How could conceptual understanding reduce the impact of spurious associations from textbook problems?

If children applied conceptual understanding of fraction arithmetic to the answers they generate, they could reject many erroneous answers based on an implausible magnitude of the answer (is it possible that 12/13 + 7/8 = 19?) or a violation of principles (is it possible that 1/2 + 1/2 = 2/4?). The process seems akin to Kahneman's (2011) discussion of the relation between Type 1 "fast" processing and Type 2 "slow" processing. Strategy choices based on spurious associations between operations and operands in textbooks seem akin to Type 1 "fast thinking." Strategy choices based on conceptual understanding that overrides the spurious associations seems akin to Type 2 "slow thinking." As Kahneman noted, the slower processing, based on conceptual understanding, can override the faster processing, based on associations.

Detecting the implausibility of answers yielded by flawed procedures could motivate children to try alternative procedures that they previously learned. The plausibility of this account was further supported by Siegler and Pyke's (2013) findings, which showed that most children who used implausible strategies on some trials used correct strategies on other, highly similar trials that differed only in the particular fractions in the problem. With repeated experience solving problems, children would seem likely to use correct strategies increasingly often, because the answers the correct strategies generate would be reinforced more often than the answers produced by the flawed strategies. Alternatively, with improved conceptual understanding, children might be able to avoid the flawed strategies entirely.

Development of whole number addition provides a useful contrastive case that illustrates how learning occurs when children possess conceptual understanding of an arithmetic operation. Even preschoolers have considerable understanding of whole number addition and subtraction (Gilmore, Göbel, & Inglis, 2018). For instance, children choose adaptively among the varied addition strategies they use, in the sense of using each approach most often on problems on which it yields favorable combinations of accuracy and speed (Siegler & Shrager, 1984). Preschoolers usually use retrieval, the fastest strategy, for problems on which it can be executed accurately. On the other hand, preschoolers usually use slower strategies, such as counting from one, on problems where those approaches are necessary for accurate performance. Such adaptive strategy choices, along with the almost total absence of implausible answers such as $3 + 4 = 2$ or $3 + 4 = 22$, seem to reflect implicit understanding of basic addition.

Preschoolers' conceptual understanding of whole number addition extends to discovery of new strategies. Siegler and Jenkins (1989) identified 4 and 5 year olds who, on a pretest, solved problems by counting from 1 but never counted-on from the larger addend, even on problems such as $2 + 9$, where counting-on from the larger addend could have been advantageous. The children were presented large numbers of addition problems, most of them involving two single-digit addends, with feedback about the answer's correctness following each problem.

Solving problems led almost all of the preschoolers to discover the counting-on strategy, though some took more than 200 problems to do so. Most children also discovered another correct strategy that was intermediate between counting from one and counting-on from the larger addend. Perhaps most striking, no preschooler ever tried a conceptually flawed strategy, such as counting the first addend twice or only counting the second addend.

Beyond this implicit understanding, young children also possess some explicit conceptual understanding of whole number addition. On the trial where they discovered the counting-on strategy, some preschoolers in Siegler and Jenkins (1989) explicitly noted its superiority to counting-from-one because, as one child put it, when you count-on, "You don't have to count a very long way." Moreover, when kindergartners in another study were asked to judge whether a strategy that an experimenter demonstrated was "very smart," "kind of smart," or "not smart," they judged counting-on, which they had not used on the pretest, to be much smarter than the conceptually flawed strategy of counting the first addend twice, which they also had not used (Siegler & Crowley, 1994).

A major difference between whole number and rational number arithmetic is that in at least some areas of whole number arithmetic, children employ a goal sketch that reduces use of flawed strategies. Goal sketches are domain-specific mechanisms for evaluating the legitimacy and usefulness of strategies in that domain. They include requirements for correct strategies as well as means for evaluating the plausibility of answers. A goal sketch for fraction multiplication might include the information that multiplying two positive fractions below one must result in an answer less than either multiplicand; any strategy that violated that principle would be rejected. A goal sketch that included such information would allow children to reject $3/5$ as a potential answer to $3/5 \times 1/5$, because that answer would be larger than one of the operands and equal to the other. Such evaluations could lead children to turn to the other main fraction multiplication strategy they know, the correct strategy, and thereafter choose it increasingly, because it produced answers that met the requirements of the goal sketch and received reinforcement.

Consistent with this view, children learn whole number arithmetic despite the presence of imbalances in the distribution of problems that children receive. For instance, children hear problems of the form 1 + n less often than problems of the form n + 1 (Siegler & Shrager, 1984). Similarly, frequency of single-digit multiplication problems in second and third grade math workbooks is correlated with accuracy on the problems (Siegler, 1988). Yet, most children eventually become proficient with addition and multiplication of whole numbers (e.g., Moore & Ashcraft, 2015), displaying little difficulty even with problems that appear relatively infrequently.

One reason why most children eventually succeed in learning at least single-digit whole number arithmetic may be that their understanding of whole number arithmetic is grounded in their knowledge of counting. Children initially use counting to generate or enumerate whole number quantities (e.g., Le Corre & Carey, 2007; Wynn, 1992), but later also count to calculate sums of whole numbers. Thus, to calculate 3 + 2, a child might count “1,2,3,4,5” or “3,4,5.” Children spontaneously discover increasingly advanced strategies for adding by counting (Siegler & Jenkins, 1989; Svenson & Sjöberg, 1983), culminating in relatively consistent retrieval of arithmetic facts from memory.

6 | PROMOTING CONCEPTUAL UNDERSTANDING AS A KEY TO EFFECTIVE INTERVENTIONS

Providing children with a foundation for understanding fraction arithmetic, analogous to the foundation that counting provides for whole number arithmetic, could greatly improve their ability to learn fraction arithmetic procedures. Several recent studies have tested experimental interventions designed with these goals in mind and have shown promising results (Braithwaite & Siegler, 2020; Dyson, Jordan, Rodrigues, Barbieri, & Rinne, 2018; Fuchs et al., 2013). Below, one of these interventions is described in detail. Our goal is not to advocate for a particular intervention, but to illustrate how children's understanding of fraction arithmetic might be improved and how doing so might facilitate children's learning of fraction arithmetic procedures.

Putting Fractions Together (PFT) is a conceptual framework that was created to help children connect their knowledge of individual fractions and fraction arithmetic, thereby improving their understanding of both (Braithwaite & Siegler, 2020). PFT uses the idea of concatenating (i.e., putting together) unit fractions to conceptualize individual fractions and sums of fractions with both equal and unequal denominators. Thus, 3/9 is conceptualized as the result of concatenating three 1/9s; 2/9 + 1/9 is conceptualized as the result of concatenating two 1/9s with one 1/9; and 3/9 + 1/6 is conceptualized as the result of concatenating three 1/9s with one 1/6. The idea of concatenating unit fractions was intended to serve as a foundation for understanding individual fractions and fraction addition in the same way that counting serves as a foundation for understanding individual whole numbers and whole number addition. Counting and PFT both involve iterating units (wholes or unit fractions) to generate quantities, but unlike counting, PFT permits combining units of different sizes (i.e., different unit fractions, as when adding 1/5 + 1/6).

Braithwaite and Siegler (2020) tested the PFT framework by conducting two intervention experiments, the first with 63 fourth and fifth graders, the second with 104 fifth and sixth graders. Children in Experiment 1 were recruited from a school serving a 95% Caucasian study body; children in Experiment 2 were recruited from two schools, both of which served more racially diverse student bodies (68% Caucasian, 23% African-American, 8% biracial, and 1% Latinx in one school, and 52% Caucasian, 28% African-American, 19% Latinx, and 2% Other in the other).

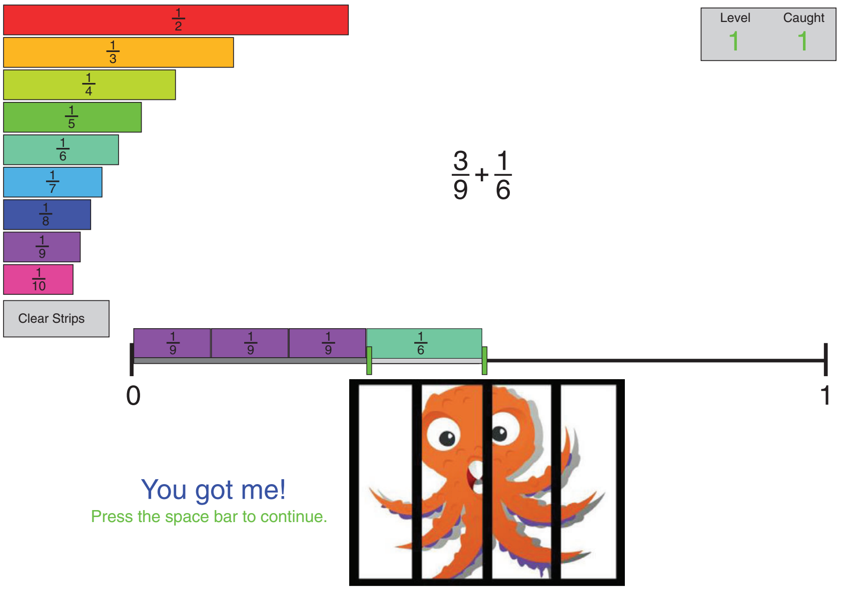

FIGURE 1 An example trial from the computer game used in the fraction intervention, involving a fraction sum. The example displays the feedback children received after "catching the monster"

The experimental designs in both experiments involved an intervention group and an active control group. All children in both conditions participated in computer game-based activities.

In the intervention condition, children created graphical representations of individual fractions and fraction sums by concatenating virtual manipulatives representing unit fractions, then used these graphical representations to estimate the magnitudes of the fractions and fraction sums on a number line in order to catch a hidden monster (Figure 1). After children played the game over multiple trials, the virtual manipulatives were withdrawn, and children continued to practice estimating fractions and fraction sums. This transition was intended to encourage children to internalize and mentally simulate the process of concatenating unit fractions.

In the active control group in Experiment 1, children received the same procedure as peers in the intervention group for estimation of the magnitudes of individual fractions. These children did not receive practice on estimating the magnitudes of the sums, but received a greater number of problems involving estimation of the magnitudes of individual fractions, thus allowing time on task to be equated for the two conditions. In the active control group in Experiment 2, children received the same procedure as in the intervention group except that they estimated whole numbers and sums of whole numbers on a 0–1,000 number line instead of fractions and fraction sums on a 0–1 number line.

Different versions of the intervention were tested in two experiments with fourth, fifth, and sixth graders. Before and after the intervention, children completed assessments requiring them to estimate the magnitudes of individual fractions and sums of fractions on a number line without using (or seeing) any virtual manipulatives. Children displayed large improvements in accuracy on these tasks from pretest to posttest, especially on the most difficult task, estimation of unequal-denominator sums (Cohen's *d*s for this task ranged from 1.52 to 1.72). Children's performance also improved on a transfer task, comparing fraction sums to 1, that they did not practice during the intervention ($d = 0.69$). Improvements on all tasks involving fraction sums were larger than those observed in the active control groups. The improvement on the transfer task was accompanied by increased use of estimation strategies, such as, “2/10 and 1/8 are less than 1/2 so that means that [their sum is] going to be less than 1.” Thus, the intervention helped children learn how to estimate sums of fractions accurately without relying on exact calculation procedures.

The development of the estimation strategies mentioned above is critical because estimation provides a method for discriminating between plausible and implausible answers without already knowing the correct procedure. For instance, a child who knows—without calculating—that 2/3 + 3/5 is greater than 1 could recognize that 2/3 + 3/5 = 5/8 is impossible. After a number of such experiences, the child might recognize that separately adding numerators and denominators cannot be a correct procedure (because it always yields impossible answers—ones no greater than at least one of the addends), and might try a correct procedure instead. Thus, improving children's ability to estimate could provide them with a conceptual basis for choosing between correct and incorrect procedures.

Returning to the issue with which this section began, how might interventions such as PFT enable children to resist the influence of biased input on learning? When deciding how to solve a problem, children without a conceptual foundation for understanding fraction arithmetic have no way to distinguish correct procedures from incorrect ones. This leaves children open to the influence of biased input. Conceptual understanding, however, could save students from relying on such experiences alone. As the above example illustrates, students with reasonable conceptual understanding of fraction arithmetic operations can use that understanding to help decide how to solve problems and to evaluate the results of their decisions. This can help them avoid errors that are currently pervasive in fraction arithmetic. PFT represents a promising approach to inculcating the conceptual understanding needed to make this possible.

ACKNOWLEDGMENTS

The authors want to thank for research support the Institute of Education Sciences, US Department of Education, through Grant R305A180514 to Columbia University; the Siegler Center for Innovative Learning and Advanced Technology Center, Beijing Normal University; and National Science Foundation Grant No. 1844140 to David W. Braithwaite and Robert S. Siegler. Any opinions, findings, conclusions, and recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the agencies that supported the research.

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How to cite this article: Siegler, R. S., Im, S.-H., & Braithwaite D. (2020). Understanding development requires assessing the relevant environment: Examples from mathematics learning. *New Directions for Child and Adolescent Development*, 1–18. https://doi.org/10.1002/cad.20372