Learning and Instruction 68 (2020) 101347

Contents lists available at ScienceDirect

Learning and Instruction

journal homepage: www.elsevier.com/locate/learninstruc

Distinguishing adaptive from routine expertise with rational number arithmetic

Jake McMullen[a,*], Minna M. Hannula-Sormunen[a], Erno Lehtinen[a,c], Robert S. Siegler[b]

[a] University of Turku, Finland
[b] Columbia University and the Siegler Center for Innovative Learning, Advanced Technology Center, Beijing Normal University, China
[c] Vytautas Magnus University, Lithuania

ARTICLE INFO

Keywords:
Adaptive expertise
Flexibility
Rational numbers
Mathematical development
Adaptive number knowledge

ABSTRACT

Adaptive expertise is a valued, but under-examined, feature of students' mathematical development (e.g. Hatano & Oura, 2012). The present study investigates the nature of adaptive expertise with rational number arithmetic. We therefore examined 394 7th and 8th graders' rational number knowledge using both variable-centered and person-centered approaches. Performance on a measure of adaptive expertise with rational number arithmetic, the arithmetic sentence production task, appeared to be distinct from more routine features of performance. Even among the top 45% of students, all of whom had strong routine procedural and conceptual knowledge, students varied greatly in their performance the arithmetic sentence production task. Strong performance on this measure also predicted later algebra knowledge. The findings suggest that it is possible to distinguish adaptive expertise from routine expertise with rational numbers and that this distinction is important to consider in research on mathematical development.

1. Distinguishing adaptive from routine expertise with rational number arithmetic

Rational number knowledge is a linchpin of students' mathematical development (Booth & Newton, 2012; DeWolf, Bassok, & Holyoak, 2015; Hurst & Cordes, 2018; Siegler et al., 2012). It is both a key outcome of early mathematical development (Steffe & Olive, 2010) and a cornerstone for later algebra knowledge (Hurst & Cordes, 2018).

Most research on rational numbers has examined students' difficulties (e.g. Jordan et al., 2013; Van Hoof, Janssen, Verschaffel, & Van Dooren, 2015); far fewer studies have examined high-level performance with basic rational number topics, such as understanding of magnitudes of individual rational numbers and standard, frequently practiced, rational number arithmetic procedures. This is unfortunate, because routine expertise in a mathematical topic often is insufficient for future success in applying the knowledge to novel situations, regardless of the strength of the routine knowledge in typical situations (Baroody, 2003; Hatano & Oura, 2012). Students less often acquire adaptive expertise, which requires more malleable, fluid knowledge that is readily applicable to novel situations. This is unfortunate, because adaptive expertise is expected to be an important predictor of future success with mathematics (Lehtinen, Hannula-Sormunen, McMullen, & Gruber, 2017). Therefore, the present study investigates the nature of adaptive expertise with rational number arithmetic.

1.1. Adaptive expertise with rational number arithmetic

Theories of adaptive expertise (e.g., Baroody, 2003) make an explicit distinction between (a) static, sparsely connected knowledge that can only be applied to typical tasks, and (b) richly connected knowledge that can be flexibly applied in novel contexts. The former typifies routine expertise; the latter typifies adaptive expertise. In the present study, we attempt to integrate previous theories of conceptual and procedural knowledge (e.g. Hiebert & Lefevre, 1986) with those of adaptive expertise (e.g. Hatano & Inagaki, 1986).

Previous theories of conceptual knowledge have proposed that the degree of interconnectedness of conceptual knowledge is crucial to its quality (Hiebert & Lefevre, 1986; Schneider & Stern, 2009). However, these theories have not pursued the distinction beyond noting that students vary in the interconnectedness of the knowledge. Based on the distinction between adaptive and routine expertise, it may be fruitful to make a distinction between highly-connected conceptual knowledge and less-connected conceptual knowledge. Previous examinations of rational number sense have more generally outlined a broad range of skills and knowledge that could typify adaptive expertise with rational numbers, including useful estimation; flexible representations; mental

* Corresponding author. Department of Teacher Education, Turun Yliopisto, 20014, Finland.
E-mail address: jake.mcmullen@utu.fi (J. McMullen).

https://doi.org/10.1016/j.learninstruc.2020.101347
Received 28 June 2019; Received in revised form 17 April 2020; Accepted 28 April 2020
Available online 14 May 2020

manipulations (often calculations); application to novel, real-world situations; and numerical judgements (e.g. Markovits & Sowder, 1994; Moss & Case, 1999). However, to the best of our knowledge, no prior study has examined specific aspects of adaptive expertise with rational numbers on a large scale in relation to features of students' routine expertise. Thus, the present study is an attempt to provide data on which future theoretical distinctions in this area can be more solidly based.

A core feature of adaptive expertise that distinguishes it from routine expertise is the ability to flexibly apply knowledge to solve novel tasks. As Baroody (2003) notes, this perspective suggests that integrating strong procedural fluency with strong conceptual knowledge is needed for adaptive expertise. However, descriptions of adaptive expertise require specification of behavioral differences that follow from the theoretical distinctions (Fazio, DeWolf, & Siegler, 2016; Schneider, Rittle-Johnson, & Star, 2011; Torbeyns, Verschaffel, & Ghesquière, 2006; Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009). One attempt to specify these behaviors led to investigations of children's performance on the arithmetic sentence production task, which is proposed to measure one aspect of adaptive expertise with whole number arithmetic (McMullen et al., 2016). Substantial individual differences emerged in the quantity and mathematical complexity of the solutions students generated on this novel calculation task (McMullen et al., 2017). These differences were related to, but not entirely explained by, procedural and conceptual knowledge on common tasks (i.e. routine knowledge). However, these studies did not explicitly examine if it is possible to empirically distinguish those aspects of performance related to adaptive expertise from those related to routine expertise. Therefore, we aim to examine one aspect of adaptive expertise with rational number arithmetic, by examining performance on the arithmetic sentence production task and determining if this should be distinguished from performance on measures of routine expertise with rational number knowledge.

1.2. The present study

To examine the nature of adaptive expertise with rational number arithmetic, we employed both variable-centered and person-centered approaches (Bergman & Magnusson, 1997; Hickendorff, Edelsbrunner, McMullen, Schneider, & Trezise, 2018). In the variable-centered approach, we used Confirmatory Factor Analysis (CFA) to determine if performance on a measure of adaptive expertise can be distinguished from performance on measures of routine expertise of rational numbers. In the person-centered approach, we used Latent Profile Analysis (LPA) to identify individual students' knowledge patterns across these measures.

Students' performance on an arithmetic sentence production task similar to one used previously with whole numbers was theorized to describe one potential behavioral manifestation of adaptive expertise with rational number arithmetic (McMullen et al., 2017). On this task, participants needed to generate as many distinct correct arithmetic sentences as possible that included subsets of five numbers to produce a target number. Each item included pairs of equivalent fractions and decimals (e.g. ½ and 0.5, ¼ and 0.25), as well a whole number (e.g. 4). Previous research has found that some students adaptively move among solution strategies based on the characteristics of the problem, but other students do not (McMullen et al., 2017). Thus, we also carried out exploratory analysis describing the solutions students produced on the arithmetic sentence production task. In particular, we examined their relative use of fractions and/or decimals and multiple operations in their solutions. As well, given the equivalent fractions and decimals that can be used as numbers in these arithmetic sentences, and the importance of representational flexibility between fractions and decimals (Deliyianni, Gagatsis, Elia, & Panaoura, 2016), we also examined how often participants used mathematically equivalent, but notationally different, solutions (i.e. using both $1/2 + 1/2$ and $0.5 + 0.5$).

Several aspects of rational number knowledge typical for routine expertise with rational numbers seemed relevant for performance on the arithmetic sentence production task. These aspects were considered routine because they did not require solving novel tasks (Baroody, 2003) and are part of a foundational understanding of the rational number concept (Van Hoof et al., 2015). Knowledge of arithmetic calculation procedures with fractions and decimals was considered routine procedural knowledge that was necessary to produce correct solutions. Three aspects of routine conceptual knowledge of rational numbers also seemed relevant to performance on the arithmetic sentence production task. One was knowledge of the magnitude of rational numbers, which seemed essential for producing arithmetic sentences that would yield the target value (Bailey, Hansen, & Jordan, 2017). Another was knowledge about rational number representations (e.g. Deliyianni et al., 2016), which seemed crucial for generating equivalent fractions and decimals in number sentences. Finally, knowledge of relations among rational number operations was expected to support high level performance on the task by opening up opportunities to use multiplication and division with numbers less than one, a type of knowledge that many students lack (Lortie-Forgues, Tian, & Siegler, 2015).

To test whether students with the best performance on the arithmetic sentence production task were simply those with the most rational number knowledge, we presented a task measuring knowledge of the density of rational numbers, the knowledge that there are infinite numbers between any two rational numbers. Performance on the arithmetic sentence production task and understanding of density seemed likely to be positively correlated. However, we did not expect that density knowledge would directly contribute to this performance. Even after years of experience with rational numbers, many students do not understand the density property (Vamvakoussi & Vosniadou, 2010; Van Hoof, Degrande, Ceulemans, Verschaffel, & Van Dooren, 2018). This presumably includes students who are skillful in solving routine rational number arithmetic problems (Vamvakoussi, Van Dooren, & Verschaffel, 2012).

We also examined whether performance on the arithmetic sentence production task predicts later algebra knowledge. Rational number knowledge in general has been found to be related to algebra knowledge (Booth & Newton, 2012; DeWolf et al., 2015; Empson, Levi, & Carpenter, 2010; Hurst & Cordes, 2018), but the sources of the relation remain unspecified. Performance on the arithmetic sentence production task seemed likely to be related to algebra knowledge because algebra requires the type of flexible mathematical thinking that the arithmetic sentence production task is intended to measure (e.g. Star, 2007).

Finally, to demonstrate that relations between performance on the arithmetic sentence production task and other mathematical knowledge do not simply reflect greater motivation or skill at learning in school, we also examined relations between this performance and reading achievement. Our hypothesis was that reading knowledge is unrelated or minimally related to performance on the arithmetic sentence production task.

2. Methods

2.1. Participants

Students from the 7th and 8th grades of a school in the southeastern US ($N = 394$; 53% female; 7th grade $n = 232$) participated in the study. The population of the school was 51% white, 28% African American, 11% Hispanic, and 5% Asian; 43% of students received free or reduced lunch. All participants had parental permission to participate and gave their own assent; the ethics board of the first authors' institution approved the study, as did district and school administration. Participants completed paper-and-pencil measures of rational number knowledge in their science classrooms in January 2017.

2.2. Measures

2.2.1. Rational number conceptual knowledge

We assessed three aspects of routine conceptual knowledge of rational numbers that might be related to adaptive expertise with rational number arithmetic: magnitudes of rational numbers, operations with rational numbers, and representations of rational numbers.

2.2.2. Rational number magnitude knowledge

Rational number magnitude knowledge was assessed through two tasks: an ordering task and a number line estimation task (Schneider & Siegler, 2010; Stafylidou & Vosniadou, 2004; Van Hoof et al., 2015). The ordering task included two fraction items (e.g. “Put the numbers in order from smallest to largest”: 6/12; 5/7; 2/6), two decimal items (e.g. “Put the numbers in order from smallest to largest”: 5.89; 5.886; 6.5), and two fraction and decimal items (e.g. "Put the numbers in order from smallest to largest": 0.5; 1/4; 5/100; 0.356). Each item was scored as correct or incorrect with a maximum score of 6 for the test. Reliability was good (Cronbach's $\alpha = 0.81$).

Number line estimation was assessed on a 0–1 number line with four items (0.6, 1/5, 3/7, and 0.42), and on a 0–5 number line with four other items (11/7, 3.7, 9/2, and 0.83). Percent absolute error was used to measure accuracy on both number lines (e.g. Siegler, Thompson, & Opfer, 2009). Reliability was acceptable for these items (Cronbach's $\alpha = 0.70$).

2.2.3. Rational number arithmetic operations knowledge

Knowledge of rational number operations was measured using six items adapted from Van Hoof et al. (2015). Items tested students' knowledge of the effects of arithmetic operations with fractions and decimals (e.g. “Is the outcome of 40 × 1/3 smaller or larger than 40?“; “What is half of 1/6?“). All items were incongruent, such that reasoning based on features of arithmetic with whole numbers would lead to incorrect answers (e.g. Multiplication always makes a number bigger). Reliability was good for these items (Cronbach's $\alpha = 0.80$).

2.2.4. Rational number representation knowledge

Knowledge of rational number representations was examined via the Number Sets Test (Geary, Bailey, & Hoard, 2009). Students had 1 min to identify as many symbolic and non-symbolic representations as possible that equaled first ½ and then 0.9. Each item had fifteen alternative answers, with nine and eight correct matches per item. Correct answers added a point; incorrect answers deleted a point. Reliability was good for these items (Cronbach's $\alpha = 0.82$).

2.3. Procedural knowledge of rational number arithmetic

Participants were also asked to solve 12 fraction arithmetic problems (2/3 − 1/3; 4/7 ÷ 1/2; 3/4 × 1/5; 8 1/2 ÷ 4 1/8; 5/7 − 1/2; 1/5 + 2/3; 7/8 + 2/8; 2 3/4 + 4 1/8; 2 6/7 + 5 1/2; 5/8 ÷ 3/8; 3 2/3 − 3/4; 3/5 × 1/5) and 12 decimal arithmetic problems (1.05 × 0.2; 0.71–0.4; 0.11 + 0.7; 5.29–4.2; 3.4 + 1.02; 0.38–0.14; 0.4 + 0.2; 0.9 ÷ 0.3; 0.4 × 0.52; 0.111 × 0.097; 3.06 × 5.3; 0.84 ÷ 0.4). Answers were scored as correct or incorrect, with the maximum score for the test being 24. Reliability was good for these items (Cronbach's $\alpha = 0.89$).

2.4. Adaptive rational number knowledge

Following previous research (McMullen et al., 2016), we use the term adaptive rational number knowledge to describe students' performance the arithmetic sentence production task with rational numbers. Students had 90 s to generate as many mathematically correct arithmetic sentences as possible that produced a target number by arithmetically combining subsets of five other numbers. First, students completed a practice item with whole numbers (make 6 by combining a subset of 1, 2, 3, and 4). Individual numbers could be used repeatedly. After completing this item, the students were encouraged to ask questions about it and then were presented four test items. Each test item included two pairs of equivalent fractions and decimals (e.g. ½ and 0.5; ¼ and 0.25) and a single whole number (e.g. 4) as the numbers from which students should make the target number (e.g. 1; See Table 1). Answers were counted as correct if they were mathematically correct, only used the given numbers, and were not literal repetitions of a previous solution. Thus, mathematically similar solutions (e.g. 1/2 + 1/2 and 0.5 + .5) and inverses (e.g. 1/4 + 1/2 and 1/2 + 1/4) were counted as correct. Participants received one point for each correct arithmetic sentence. Reliability was good (Cronbach's $\alpha = 0.86$).

Table 1
Given and target numbers for arithmetic sentence production task with rational numbers.

Item	Given Numbers	Target Number
1	1/2 ; 1/4; 0.5; 0.25; 4	1
2	1/2 ; 1/8; 0.5; 0.125; 2	1/4
3	1/4; 3/4; 0.25; 0.75; 2	.5
4	3/2; 3/4; 1.5; 0.75; 2	3

To classify the solutions, we coded the number of correct responses that included (a) fractions, (b) decimals, (c) both fractions and decimals (e.g. 1/2 + 0.5), and (d) both addition/subtraction and multiplication/division (e.g. 2 * 1/4 + 1/2). The number of mathematically equivalent solutions (e.g. 1/2 + ½, 0.5 + 0.5, 0.5 + 1/2, and 1/2 + .5) used by a participant within an item was also coded. As these codings did not require subjective interpretations, they were not checked for inter-rater reliability.

2.5. Rational number density knowledge

Density knowledge was assessed using short-answer and multiple-choice items (Vamvakoussi & Vosniadou, 2010). It was not included in the main analyses, because it was not considered directly relevant to adaptive rational number knowledge. Rather, it was used as a control for ability to learn difficult, and often implicitly taught, mathematical content. The 10 open-ended or multiple-choice items asked students, for example: “Are there other fractions/decimals between [5/7 and 6/7] OR [0.3 and 0.4]? If so, how many?“.

Each response was scored as indicating full (3 points), partial (2 points), limited (1 point), or incorrect (0 points) knowledge of density:

- *Full knowledge* responses displayed a mathematically correct concept of the density of rational numbers (e.g. There are an infinite number [of numbers between 3/7 and 4/7.]).
- *Partial knowledge* responses expressed a partially correct understanding of the dense nature of rational numbers but were constrained by the representation (e.g. there are an infinite number of fractions between 3/7 and 4/7 [but not an infinite number of decimals]).
- *Limited knowledge* answers expressed an understanding that there are many, but not unlimited, numbers between any two other rational numbers.
- *Incorrect knowledge* answers displayed no understanding of the density of rational numbers, for example stating that there are only one or no number(s) between any two other numbers.

Reliability was high for the density items (Cronbach's $\alpha = 0.93$)

2.6. Algebra and reading achievement

End-of-year statewide standardized test scores for mathematics and reading were obtained. Tests were taken four months after the rational number knowledge assessment. Only a sub-set of the original sample completed the algebra test, because students completed different mathematics tests based on which course they took that year: general mathematics, algebra, or geometry. Since these topics were not directly comparable (e.g. the algebra test was more difficult than the general mathematics tests), and we were particularly interested in the relation between rational number knowledge and algebra, only algebra scores were examined. Scores were obtained for the 127 students who completed the algebra test (highest possible score = 6) and the 352 students who completed the reading test (highest possible score = 5).

2.7. Analysis

All analyses were carried out in Mplus version 8.0 (Muthén & Muthén, 1998-2017). CFA was used to examine the hypothesis that adaptive expertise with rational number arithmetic is distinct from routine expertise with rational numbers. Two manifest variables for each knowledge type were used in the analysis (a) the odd (i.e. Items 1 and 3): and even (i.e. Items 2 and 4) items from the arithmetic sentence production task for adaptive number knowledge, (b) fraction and decimal arithmetic procedural scores for procedural knowledge, (c) fraction and decimal ordering and number line estimation for magnitude knowledge, (d) fraction and decimal operations knowledge, and (e) fraction and decimal representation knowledge. Overall model fit was evaluated based on thresholds of less than 0.05 for root-mean-square error of approximation (RMSEA), greater than 0.95 for comparative fit index (CFI), greater than 0.90 for the Tucker-Lewis Index (TLI) and less than 0.08 for the standardized root-mean-square residual (SRMR). Lower values for Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) indicate a better fit when comparing models (Hu & Bentler, 1999).

Latent Profile Analysis (LPA) was used to examine patterns of rational number knowledge. The estimation method was maximum likelihood with robust routine errors, which is a full information approach that can handle missing-at-random data. The analyses of LPA models were carried out as mixture models, in which 1000 and 100 random start values were used in the first and second steps of model estimation, respectively, to ensure the validity of the solution (Geiser, 2013). Model fits were evaluated with a combination of statistical indicators and substantive theory to determine the most suitable number of latent classes and best fitting models (Nylund, Asparouhov, & Muthén, 2007). Entropy values that approach 1 signify more certainty in the resulting classification. Finally, a significant result of the Parametric Bootstrapped Likelihood Ratio Test (BLRT) and Lo-Mendell-Rubin (LMR) test suggests support for the k-class solution in comparison with the k-1-class solution. In other words, a significant result on this test suggests that the use of that model is more appropriate than the model that has one less class in it.

3. Results

Table 2 details the means and standard deviations for all rational number knowledge measures, as well as the correlations among the measures. As can be seen in Table 2, the measures of rational number knowledge were moderately to strongly interrelated.

3.1. Distinguishing adaptive rational number knowledge from routine procedural and conceptual knowledge

To determine whether adaptive rational number knowledge is distinct from routine knowledge of rational numbers, we tested whether the constructs were better modeled separately or combined in a series of confirmatory factor analyses. If adaptive rational number knowledge is more appropriately modeled separately from routine procedural and conceptual knowledge of rational numbers, this would provide evidence of discriminant validity.

Table 3 describes the CFA fit statistics for the series of models, including a unitary one-factor model (Model A) with all variables included in the same factor; three different two-factor models that combine adaptive and routine conceptual, and routine procedural knowledge into pairs of types of knowledge: (Model B) adaptive and all conceptual, (Model C) adaptive and procedural, or (Model D) procedural and all conceptual; the three-factor model (Model E) that treats adaptive, procedural, and all conceptual knowledge as separate factors; four different four-factor models that separate out the conceptual components and pairs adaptive knowledge with each of the four knowledge components leaving the others separate: (Model F) adaptive and procedural, (Model G) adaptive and magnitude, (Model H) adaptive and operations, and (Model I) adaptive and representations; and the five-factor model (Model J) that treats all five aspects of knowledge as separate latent variables.

The fits of the estimated models suggest that either a three-factor model, which distinguishes among adaptive and routine procedural and conceptual rational number knowledge, or the five-factor model, which distinguishes among all five aspects of knowledge, is most appropriate. This suggests that adaptive rational number knowledge is best modeled as a construct distinct from routine conceptual and procedural knowledge.

3.2. Profiles of rational number knowledge

To further examine whether adaptive rational number knowledge is a unique aspect of students' understanding of rational numbers, we conducted an LPA with the five aspects of rational number knowledge as indicators: 1) *magnitude knowledge* (sum of standardized scores for ordering and number line estimation; r (394) = 0.63), 2) *operations knowledge*, 3) *procedural knowledge of arithmetic*, 4) *representation knowledge*, and 5) *adaptive rational number knowledge*. The three aspects of routine conceptual knowledge (magnitude, operations, and representations) were used as separate indicators in the LPA modelling. This was done to determine if there were varying relations among these variables, and in particular whether adaptive rational number knowledge was differently related to the different aspects of routine rational number knowledge. This decision is in line with the results of the CFA suggesting that the 5-factor model J and 3-factor model E are similarly appropriate for describing rational number knowledge.

Table 4 details the fit indices for the two-through seven-class solutions for the LPA. These model fit indices suggested that the four-class model was most appropriate, as the BIC was lowest with this model, and both BLRT and VLMR tests suggested that the five-class model was not better than the four class model. Entropy for this model was sufficient (i.e. > 0.6; Collins & Lanza, 2010), and the posterior probabilities for the classes showed that the model had high agreement with regard to placing most individuals clearly into a particular class (all probabilities ≥ .87).

Fig. 1 details the means for each indicator for the different latent classes, allowing for a comparison of rational number knowledge of students in each class. Labels were assigned to the latent classes based on our interpretation of the quantitative results. The 26% of children in the *Basic* class had relatively low rational number knowledge of all types. The 28% of children in the *Procedural* class had above-average performance on arithmetic procedural knowledge problems, but below-average performance on magnitude and operation conceptual knowledge, representation knowledge, and adaptive rational number knowledge problems. The 35% of children in the *Routine* class had relatively well-developed rational number knowledge of all types, including adaptive rational number knowledge. Finally, the 10% of children in the *Adaptive* class performed similarly to those in the Routine Expertise class on four of the five measures but performed much better than students in any of the other four profiles on the measure of adaptive knowledge, the arithmetic sentence production task.

Table 2
Means of and correlations among rational number knowledge measures for the whole sample and only in Routine and Adaptive profiles. Means and standard deviations of the proportion of correct responses of each variable are shown in Column 2. Correlations among the variables are shown in Columns 3–6.

A Whole sample (N = 394)	M (SD)	1. Magnitude	2. Operations	3. Procedural	4. Representations
1. Magnitude knowledge					
a. Ordering	.46 (.33)				
b. Number line Estimation (PAE)	16.61 (10.34)				
2. Operation conceptual knowledge	.45 (.34)	.76***			
3. Arithmetic procedural knowledge	.22 (.12)	.71***	.69***		
4. Representation knowledge	.62 (.28)	.75***	.71***	.65***	
5. Adaptive rational number knowledge (total correct)	9.05 (7.21)	.71***	.71***	.68***	.69***
B Routine and Adaptive profiles (n = 183)					
1. Magnitude knowledge					
a. Ordering	.70 (.23)				
b. Number line Estimation (PAE)	10.18 (7.17)				
2. Operation conceptual knowledge	.72 (.20)	.35***			
3. Arithmetic procedural knowledge	.58 (.17)	.27**	.26**		
4. Representation knowledge	.79 (.13)	.25**	.26**	.29***	
5. Adaptive rational number knowledge (total correct)	12.27 (3.74)	.13	.13	.19*	.22*

Notes: PAE = percent absolute error. *$p < .05$; **$p < .01$; ***$p < .001$.

Table 3
Descriptions and fit statistics for estimated Confirmatory Factor Analyses.

Model	Type	RMSEA (< .05)	CFI (> .95)	TLI (> .90)	SRMR (< .08)	AIC	BIC	X^2 (df)
A	1 factor (Adaptive + Procedural + All Conceptual)	.085	.94	.93	.032	19591	19734	206(54)***
B	2 factor (Adaptive + All Conceptual, Procedural)	.077	.95	.94	.029	19560	19707	175(53)***
C	2 factor (Adaptive + Procedural, All Conceptual)	.073	.96	.95	.030	19547	19694	163(53)***
D	2 factor (Adaptive, Procedural + All Conceptual)	.062	.97	.96	.027	19517	19665	133(53)***
E	3 factor (Adaptive, Procedural, All Conceptual)	.048	.98	.98	.023	19484	19639	98(51)***
F	4 factor (Adaptive + Procedural, Magnitude, Operations, Representations)	.065	.97	.96	.026	19522	19689	129(48)***
G	4 factor (Adaptive + Magnitude, Procedural, Operations, Representations)	.069	.96	.95	.027	19530	19697	137(48)***
H	4 factor (Adaptive + Operations, Procedural, Magnitude, Representations)	.066	.97	.96	.025	19523	19690	130(48)***
I	4 factor (Adaptive + Representations Procedural, Magnitude, Operations)	.073	.96	.95	.027	19542	19709	149 (48)***
J	5 factor (Adaptive, Procedural, Magnitude, Operations, Representations)	.032	.99	.99	.018	19461	19644	62 (44)*

Table 4
Fit measures of latent profile models.

Number of Classes	AIC	BIC	Entropy	BLRT (p)	VLMR (p)
2	4639	4743	.86	< .001	< .001
3	4561	4689	.85	< .001	.02
4	4500	4651	.83	< .001	.09
5	4492	4668	.83	.10	.45
6	4473	4671	.90	< .001	.66
7	Did not converge				

Patterns of routine procedural, routine conceptual, and adaptive rational number knowledge varied across the four profiles. Procedural knowledge was fairly strong in the Procedural, Routine, and Adaptive profiles, with little distinction among these profiles. Routine conceptual knowledge of rational number magnitudes, operations, and representations was relatively low among children in the Basic and Procedural profiles, whereas children who fit the Routine and Adaptive profiles had similar high levels of routine conceptual knowledge. Adaptive rational number knowledge varied most widely among the profiles. Students who fit the Basic and Procedural profiles had similarly low levels of adaptive number knowledge; peers who fit the Routine profile had somewhat higher adaptive knowledge; those who fit the Adaptive profile clearly outperformed children in all three other profiles. There appeared to be a hierarchy in the three types of knowledge that suggested that: (a) procedural knowledge can exist without routine conceptual knowledge, but the reverse does not occur; and (b) routine conceptual knowledge is necessary, but not sufficient for exceptional adaptive number knowledge.

3.3. Describing adaptive rational number knowledge

To explore the features of students' responses that might explain inter-individual differences in adaptive number knowledge, we compared performance on the arithmetic sentence production task of children who differed in their most likely class membership in the LPA. The most likely profile membership distributions did not differ by grade (χ^2 (3) = 6.27, p = .10).

Table 2 reveals the pattern of correlations between the different aspects of rational number knowledge among the students in the Routine and Adaptive profiles. Due to the nature of profile formation with LPA (e.g. Hickendorff et al., 2018), the relations were much weaker than in the sample as a whole. This reflects a kind of attenuation of range, because the variation within each of the profiles (and across the top two profiles for most indicators) is much less than the variation in the total data set.

One-way ANOVAs were run to examine group differences in the proportion of responses on the arithmetic sentence production task that included (a) fractions, (b) decimals, (c) both fractions and decimals, (d) multiple arithmetic operations, and (e) mathematically equivalent solutions (e.g. 1/2 + 1/2 is equivalent to 0.5 + 0.5 and 0.5 + 1/2).

As shown in Table 5, the most substantial differences among students who best fit different profiles were in use of both fractions and decimals in a single solution and in use of mathematically equivalent solutions within an item. Post-hoc comparisons of performance of children who best fit the Routine and Adaptive classes showed that these children differed only in the proportion of solutions that were mathematically equivalent (mean difference = 0.12, SE = 0.03, $p < .001$, Cohen's d = 0.73). Children who best fit the Routine Expertise profile used equivalent solutions on about 1/4 of trials, whereas children who best fit the Adaptive profile used mathematically similar

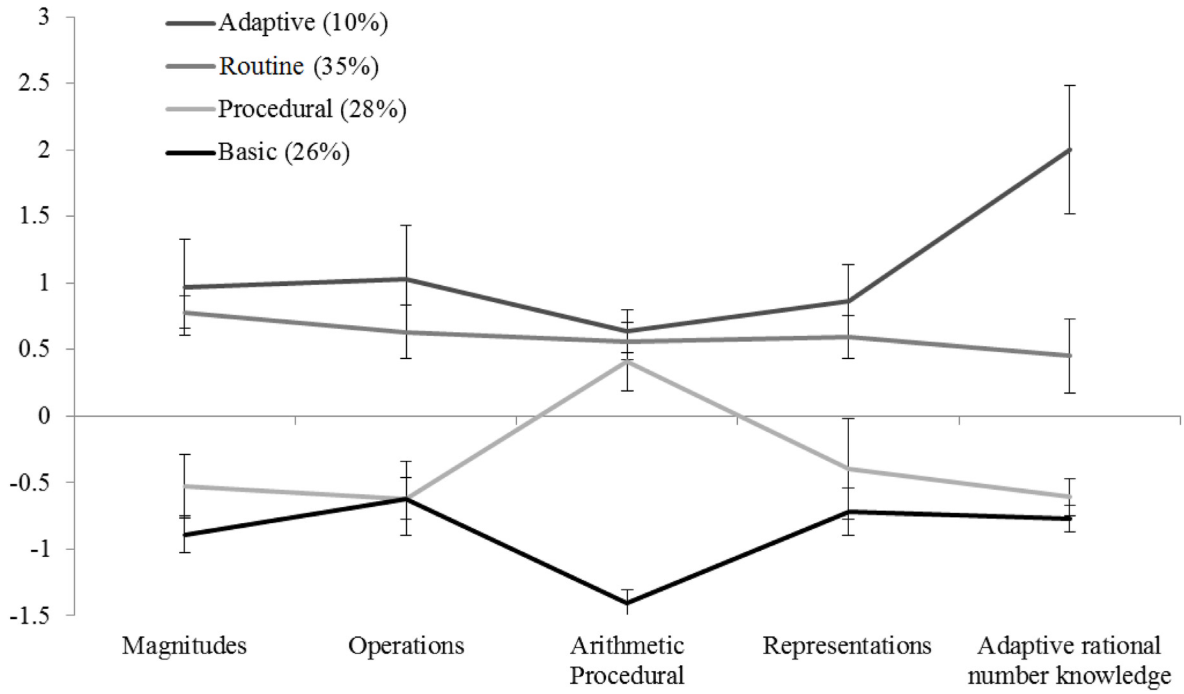

Fig. 1. Mean scores for latent classes for each aspect of rational number knowledge. Error bars: ± 2 *S.E.*

solutions on about 1/3 of trials.

We also compared the frequency of the most common, mathematically unique, solutions among students in the four profiles (Table 6). The frequency of correct, mathematically-unique solutions followed a typical decreasing power law function. Among the 118 unique correct solutions, 5 accounted for 50% of the correct solutions and 25 accounted for almost 95%. Students who best fit the Adaptive profile (a) more often generated less common solutions and (b) generated more instances of the most common unique solutions than students who best fit the Routine Expertise profile. This pattern was consistent across problems. Thus, children who best fit the Adaptive profile displayed a more diverse set of solutions, as well as displays that more often included different representations.

Given that even within the Adaptive and Routine profiles, these 25 solutions accounted for 92% or 93% of correct solutions, differences in performance on the arithmetic sentence production task appear to stem from both the frequency of generating novel, correct solutions and also finding more versions of mathematically equivalent solutions (e.g. not only using ½ + ½, but also 0.5 + .5, ½ + 0.5, and 0.5 + ½). This analysis indicates that students in the Adaptive group were not simply relying on “tricks” to game the system, such as multiply and dividing by the same number repeatedly.

3.4. Relation to density knowledge

In order to confirm that adaptive rational number knowledge was not simply a proxy for greater rational number knowledge, we ran an ANOVA examining levels of density knowledge by most likely group membership. While we find that density knowledge (M = 9.45, SD = 7.32) and adaptive rational number knowledge we highly correlated, r (394) = 0.60, $p < .001$, we were particularly interested in examining potential differences in density knowledge between the Routine and Adaptive profiles. There were differences in rational number density knowledge across profiles, F (3, 388) = 122.77, $p < .001$, $\eta^2 = 0.49$ (see Fig. 2). Density appeared to follow the same pattern as other features of rational number conceptual knowledge, with the Basic (mean = 4.76, SD = 4.42) and Procedural (mean = 4.69; SD = 4.64) profiles having similarly low levels, and the Routine (mean = 14.52, SD = 5.76) and Adaptive (mean = 16.25, SD = 6.83) profiles having similarly high levels. The Routine and Adaptive profiles did not significantly differ from each other (mean difference = 1.73, p = .26, Cohen's d = 0.29). These results suggest that density knowledge is more closely aligned with other aspects of routine conceptual knowledge than with adaptive rational number knowledge per se.

Table 5

Group means, standard errors (in parentheses) and ANOVA-test values for the proportion of solutions using Fractions, Decimals, both Fractions and Decimals, both addition/subtraction and multiplication/division (Multiple Operations), and mathematically equivalent solutions.

	Basic (n = 93)	Procedural (n = 117)	Routine Expertise (n = 141)	Adaptive (n = 40)	Total	F (3, 387)	p	η^2
Fraction	.47 (.04)	.40 (.03)	.57 (.01)	.57 (.02)	.50 (.01)	10.24	< .001	.07
Decimal	.56 (.05)	.50 (.03)	.61 (.01)	.58 (.02)	.56 (.02)	4.76	.003	.04
Fraction and Decimal	.08 (.02)	.06 (.01)	.23 (.02)	.20 (.02)	.14 (.01)	22.77	< .001	.15
Multiple Operations	.09 (.03)	.12 (.02)	.19 (.01)	.26 (.02)	.15 (.01)	7.13	< .001	.05
Equivalent Solution	.09 (.02)	.11 (.01)	.24 (.01)	.35 (.02)	.18 (.01)	40.86	< .001	.24

Table 6
Most common mathematically unique solutions and their frequency of use overall and by rational number knowledge group. Decimal representation of solution is shown, but all instances of mathematically equivalent solutions using fractions and/or decimals are included in the counts.

Item	Decimal representation of solution	Total Number of instances	Average number of correct solutions				Ratio Adaptive to Traditional
			Adaptive	Routine	Procedural	Basic	
3	.75–.25	431	2.5	1.7	0.5	0.4	1.47
1	.5 + .5	374	1.8	1.3	0.6	0.5	1.38
1	.25*4	333	1.7	1.2	0.6	0.3	1.42
4	1.5 + 1.5	264	1.6	0.9	0.3	0.4	1.78
3	.25*2	237	1.6	0.9	0.2	0.2	1.78
3	.25 + .25	225	1.4	0.8	0.2	0.2	1.75
4	1.5*2	222	1.4	0.7	0.3	0.3	2.00
1	.25 + .25+.25 + .25	185	1.2	0.5	0.3	0.2	2.40
2	.125*2	144	1.2	0.6	0	0.1	2.00
2	.125 + .125	113	1.3	0.3	0.1	0.1	4.33
1	.25 + .5+.25	113	0.9	0.5	0	0	1.80
2	.5:2	104	0.8	0.4	0.1	0	2.00
2	.5*.5	93	0.6	0.2	0.1	0.2	3.00
2	.125:.5[a]	89	0.2	0.1	0.3	0.3	2.00
4	.75 + .75+.75 + .75	63	0.9	0.1	0.1	0	9.00
4	.75 + .75+1.5	45	0.4	0.2	0.1	0	2.00
1	4:4; .5:.5; .25:.25	37	0.5	0.1	0.2	0.2	5.00
1	4*.5*.5	28	0.2	0.1	0	0	2.00
1	.5*2	26	0.2	0.1	0.1	0	2.00
4	2*2*.75	24	0.2	0.1	0	0	2.00
3	2-.75-.75	24	0.2	0.1	0	0	2.00
4	.75*2 + 1.5	17	0.3	0	0	0	N/A
2	.5-.125-.125	14	0.1	0	0	0	N/A
3	(.25 + .75):2	12	0.2	0	0	0	N/A
	Sum	3264	21.40	10.90	4.10	3.40	
	Proportion of Total Responses	.93	0.92	0.92	0.95	0.94	

[a] The most common instance of this answer was 1/8:1/2, which was both mathematically correct but also follows natural number conventions (i.e. 8:2 = 4). This may explain the relatively high use of this solution among the less advanced groups.

3.5. Predicting algebra and reading

We examined how LPA profile membership predicted performance on end-of-year statewide standardized tests for algebra and reading. ANOVAs revealed that performance across profiles differed on both tests: Algebra, F (3, 119) = 24.7, $p < .001$, $\eta^2 = 0.38$; Reading, F (3, 344) = 78.4, $p < .001$, $\eta^2 = 0.41$. Most important, children who best fit the Adaptive profile had higher Algebra scores than those who best fit the Routine expertise profile (mean difference = 0.91, p = .001, Cohen's d = 0.93), but children who best fit the Adaptive and Routine expertise profiles did not differ in their Reading scores (mean difference = 0.46, p = .14, Cohen's d = 0.47).

The inclusion of only the sub-set of the sample that completed the algebra test as a part of the year-end state-wide testing may have affected these results. However, the Chi-square test of routine or adaptive profile membership and algebra test inclusion revealed that membership in these profiles was not related to which year-end test the students took. This indicates that there was no interaction between sub-sample of algebra test takers and membership in the Routine versus Adaptive profiles and suggests that the results of differences in algebra knowledge on the year-end test was not due to sampling effects. Additionally, the pattern of reading performance found in the initial analysis was replicated in the sub-sample of algebra students, further confirming apparent lack of sampling effects.

4. Discussion

Adaptive expertise with rational number arithmetic appears to be distinguishable from routine expertise with rational numbers. This is confirmed by both variable-centered and person-centered approaches. CFA indicated that the most appropriate models for the different aspects of rational number knowledge treat performance on a measure of adaptive expertise as separate from performance on measures of routine expertise. Additionally, the person-centered approach revealed that

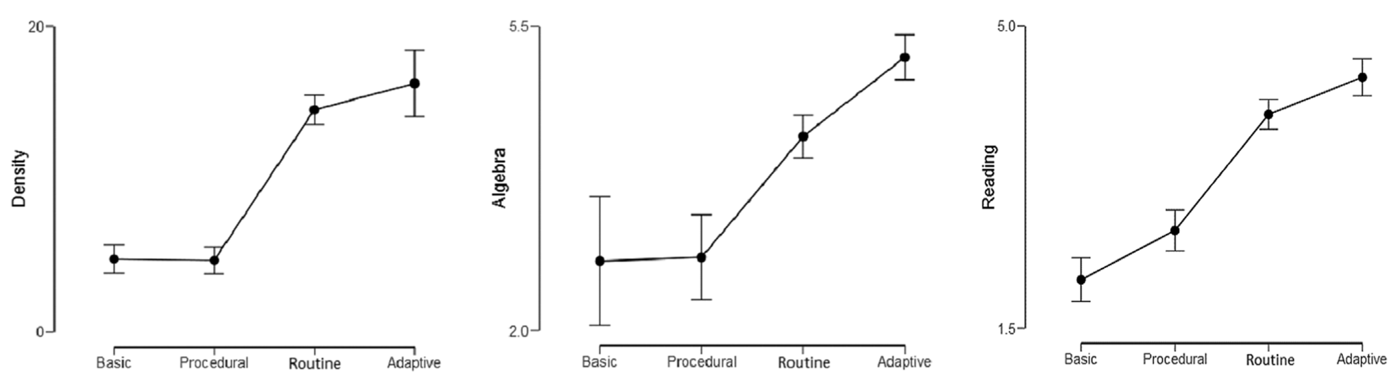

Fig. 2. Mean scores for density, algebra, and reading by latent profile. Error bars represent 95% confidence intervals. Density: n = 394; Algebra: n = 126; Reading: n = 352.

there is substantial variation in performance on the arithmetic sentence production task among those students who performed well on measures of routine procedural and conceptual knowledge. Adaptive expertise with rational number arithmetic does not require extraordinary levels of general school (i.e. reading) achievement nor exceptional rational number knowledge of other types, such as a correct understanding of density. However, such adaptive expertise is useful, being closely related to algebra learning, perhaps because it reflects the well-connected knowledge of numerical characteristics and arithmetic relations among rational numbers.

The distinction between adaptive and routine expertise with rational number arithmetic appears most relevant at higher levels of performance. Strong routine conceptual and procedural knowledge of rational numbers appears necessary, but not sufficient, for adaptive expertise with rational number arithmetic. For three of the four profiles that emerged in the LPA, performance on the arithmetic sentence production task paralleled routine conceptual knowledge of rational numbers. However, this relation was much weaker when looking within the top 45% of students, all of whom had high levels of routine procedural and conceptual knowledge. The weaker relation among those students who were proficient in routine procedural and conceptual knowledge suggests that adaptive expertise with rational number arithmetic may not simply reflect greater routine knowledge.

These results suggest that strong performance on the arithmetic sentence production task reflects a particular behavioral manifestation of the theoretical construct of adaptive expertise and, as such, requires well-integrated conceptual and procedural knowledge about rational numbers that is readily applicable in novel situations. However, the present study provides too little detailed insight about the cognitive processes needed to solve the arithmetic sentence production task to conclude if a new construct is justified to describe performance on this task or if the current definitions of advanced conceptual knowledge (e.g. Hiebert & Lefevre, 1986; Schneider & Stern, 2009) cover the nature of knowledge applied in these tasks. As well, further research is needed to clarify if there are multiple distinct aspects of adaptive expertise with rational number arithmetic, as is suggested by previous research on adaptive expertise (e.g. Torbeyns et al., 2006). If so, it may be useful to describe performance on the arithmetic sentence production task using a separate construct. In this case, we propose the term adaptive rational number knowledge.

These results enrich understanding of adaptive expertise. While theoretical accounts suggest that one requirement of adaptive expertise is the integration of conceptual and procedural knowledge (e.g. Baroody & Rosu, 2004), our results indicate that the interconnectedness between different conceptual features may also be crucial for high-level adaptive expertise. Students often struggle to use conceptual knowledge of one type (e.g. fraction magnitudes) in solving tasks of another (e.g. fraction arithmetic), even when highly relevant (Braithwaite & Siegler, 2020). Our results show that even among those students with strong conceptual knowledge, there are differences in the frequency with which these students can combine this disparate conceptual knowledge into a coherent procedural process to solve a novel task. To do well on the arithmetic sentence production task, students needed to integrate knowledge of rational number representations, rational number arithmetic, and rational number magnitudes into a single procedural process. Although some solutions may have reflected memorized facts (e.g. $1/2 + 1/2 = 1$), high level performance would have required students to combine their knowledge of rational number magnitudes, representations, and operations – for example knowing that 0.25 is less than 1, but that $4 * 0.25 = 1$, or by recognizing that $1/4 + 1/4$, $0.25 + .25$, $1/4 + .25$ will all yield 1/2. Further investigation of the interplay between the procedural processes and conceptual networks involved in the task would better illustrate how these different knowledge components contribute to success on the task (Rittle-Johnson & Siegler, 1998).

These findings suggest that the arithmetic sentence production task may be informative for measuring skills and knowledge that extend beyond traditional measures of rational numbers, even measures of advanced conceptual knowledge of rational numbers (e.g. Van Hoof et al., 2015). These traditional measures of routine conceptual knowledge explicitly guide the individual to the relevant knowledge needed to solve the task. In contrast, the arithmetic sentence production task required meeting novel constraints to generate arithmetic combinations of measures that yield specific magnitudes. Thus, the task assesses a different type of knowledge than, for example, determining the number of numbers between two rational numbers (Vamvakoussi & Vosniadou, 2010).

Students in the two highest performing profiles (i.e. Routine and Adaptive Expertise) did not substantially differ in reading achievement or density knowledge, suggesting that the differences in these profiles are not a matter of general school success or mathematical precociousness. This is consistent with previous descriptions of adaptive expertise as not merely indicating stronger routine expertise (Baroody, 2003). However, students who best fit the two high-achieving profiles differed substantially in algebra knowledge, confirming expectations that adaptive expertise should facilitate future learning by allowing for the ready application of knowledge in novel contexts. The flexibly-applicable and well-integrated knowledge that appears to describe strong adaptive expertise with rational number arithmetic may be useful when dealing with complex numerical relations in algebra. Unfortunately, in the present study, it was not possible to examine which aspects of algebra knowledge are related to high levels of adaptive expertise with rational number arithmetic, something that should be addressed in future studies.

4.1. Limitations and future directions

To date, the arithmetic sentence production task has not been used with other measures of adaptive expertise. Despite its usefulness in capturing features of what could be described as adaptive knowledge with whole number and rational number arithmetic, relying on a single measure limits the interpretability of the construct. Generating additional measures of adaptive expertise with rational number arithmetic would strengthen the connection between theory and evidence. In particular, examining how students construct arithmetic sentence problems in more routine situations (i.e. missing value problems) and examining the relation between performance on the arithmetic sentence production task and more traditional measures of flexibility, such as procedural flexibility (e.g. Schneider et al., 2011) and mathematical creativity (e.g. Kattou, Kontoyianni, Pitta-Pantazi, & Christou, 2013) could prove fruitful. Likewise, examining whether performance on the task is specific to the types of numbers used (e.g. natural numbers, rational numbers) or if it is general across different aspects of numerical knowledge would be worthwhile.

Another limitation of the present study is its failure to specify the mental processes underlying success on the arithmetic sentence production task. Analysis of students' responses argued against the possibility that some were "gaming" the system to produce a large number of solutions. However, examining students' performance on a varying set of items (e.g. items without equivalent fractions and decimals or items with solutions based entirely on multiplication and division) might provide insight into the processes underlying performance on the task. Evidence regarding cognitive and motivational correlates, longitudinal predictors, and varied products of high level adaptive expertise with rational numbers is needed to better understand the phenomenon, especially with regard to general cognitive abilities.

5. Conclusions

In contemporary society, mathematical knowledge often needs to be flexibly applied to varied situations (Gravemeijer, Stephan, Julie, Lin, & Ohtani, 2017). The goal of producing such flexibly-applicable knowledge is highlighted in curricula in many countries (Mullis, Martin, Goh, & Cotter, 2016) and is connected to the term adaptive expertise used in learning research (e.g. Hatano & Inagaki, 1986). The present study presents a new perspective on a kind of adaptive expertise that may be useful for reaching this goal. Analyses of rational numbers should be broadened to include varying tasks types such as the arithmetic sentence production task, which are not usually part of classroom instruction but that could be used as tools to assess, adaptive expertise with rational numbers.

CRediT authorship contribution statement

Jake McMullen: Investigation, Writing - original draft, Conceptualization, Methodology, Writing - review & editing. **Minna M. Hannula-Sormunen:** Conceptualization, Methodology, Writing - review & editing. **Erno Lehtinen:** Conceptualization, Methodology, Writing - review & editing. **Robert S. Siegler:** Conceptualization, Methodology, Writing - review & editing.

Acknowledgements

We gratefully acknowledge the support of the Academy of Finland Grant 311080, the Institute of Education Sciences, US Department of Education, Grant R305A180514 to Columbia University and the Weiming Education (Hong Kong) Limited; Yu Panglin Charitable Trust (Hong Kong) and Alvin I. and Peggy S. Brown Charitable Foundation, Inc. (USA). We would like to thank all the participants, teachers, and assistants without whom this research could not have been carried out.

References

Bailey, D. H., Hansen, N., & Jordan, N. C. (2017). The codevelopment of children's fraction arithmetic skill and fraction magnitude understanding. *Journal of Educational Psychology, 109*(4), 509–519. https://doi.org/10.1037/edu0000152.

Baroody, A. J. (2003). The development of adaptive expertise and flexibility: The integration of conceptual and procedural knowledge. In A. J. Baroody, & A. Dowker (Eds.). *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 1–33). London: Lawrence Erlbaum.

Baroody, A. J., & Rosu, L. (2004). Adaptive expertise with basic addition and subtraction combinations — the number sense view. *Paper presented at the annual meeting of the American educational research association, san Francisco, CA, April 2004, as part of a symposium titled, "developing adaptive expertise in elementary school arithmetic," and chaired by A. J. Baroody & J. Torbey* (pp. 1–30). .

Bergman, L. R., & Magnusson, D. (1997). A person-oriented approach in research on developmental psychopathology. *Development and Psychopathology, 9*(2), 291–319. https://doi.org/10.1017/S095457949700206X.

Booth, J. L., & Newton, K. J. (2012). Fractions: Could they really be the gatekeeper's doorman? *Contemporary Educational Psychology, 37*, 247–253. https://doi.org/10.1016/j.cedpsych.2012.07.001.

Braithwaite, D. W., & Siegler, R. S. (2020). Putting fractions together. *Journal of Educational Psychology*. https://doi.org/10.1037/edu0000477.

Collins, L. M., & Lanza, S. T. (2010). *Latent Class and Latent Transition Analysis: With applications in the social, behavioral, and health sciences.* Hoboken, NJ: Wiley.

Deliyianni, E., Gagatsis, A., Elia, I., & Panaoura, A. (2016). Representational flexibility and problem-solving ability in fraction and decimal number addition: A structural model. *International Journal of Science and Mathematics Education, 14*, 397–417. https://doi.org/10.1007/s10763-015-9625-6.

DeWolf, M., Bassok, M., & Holyoak, K. J. (2015). From rational numbers to algebra : Separable contributions of decimal magnitude and relational understanding of fractions. *Journal of Experimental Child Psychology, 133*, 1–13. https://doi.org/10.1016/j.jecp.2015.01.013.

Empson, S. B., Levi, L., & Carpenter, T. P. (2010). The algebraic nature of fractions: Developing relational thinking in elementary school. In J. Cai, & E. Knuth (Eds.). *Early algebraization: Cognituve, curricular, and instructional perspectives*New York: Springer. https://doi.org/10.1145/3126686.3126754.

Fazio, L. K., DeWolf, M., & Siegler, R. S. (2016). Strategy use and strategy choice in fraction magnitude comparison. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 42*(1), 1–16. https://doi.org/10.1037/xlm0000153.

Geary, D. C., Bailey, D. H., & Hoard, M. K. (2009). Predicting mathematical achievement and mathematical learning disability with a simple screening tool: The number sets test. *Journal of Psychoeducational Assessment, 27*, 265–279. https://doi.org/10.1177/0734282908330592.

Geiser, C. (2013). *Data analysis with MPlus.* New York: Guilford Press.

Gravemeijer, K., Stephan, M., Julie, C., Lin, F.-L., & Ohtani, M. (2017). What mathematics education may prepare students for the society of the future? *International Journal of Science and Mathematics Education, 15*, 105–123. https://doi.org/10.1007/s10763-017-9814-6.

Hatano, G., & Inagaki, K. (1986). Two courses of expertise. In H. Stevenson, J. Azuma, & K. Hakuta (Eds.). *Child development and education in Japan* (pp. 262–272). New York: W. H. Freeman & Co. https://doi.org/10.1002/ccd.10470.

Hatano, G., & Oura, Y. (2012). Commentary : Reconceptualizing school learning using insight from expertise research. *Educational Researcher, 32*(8), 26–29.

Hickendorff, M., Edelsbrunner, P., McMullen, J., Schneider, M., & Trezise, K. (2018). Informative tools for characterizing individual differences in learning: Latent class, latent profile, and latent transition analysis. *Learning and Individual Differences, 66*. https://doi.org/10.17605/OSF.IO/JMTCV.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.). *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Hillsdale, NJ: Lawrence Erlbaum Associates.

Hu, L. T., & Bentler, P. M. (1999). Cutoff criteria for fit indices in covariance structure analysis: Conventional criteria versus new alternatives. *Structural Equation Modeling, 6*, 1–55. https://doi.org/10.1080/10705519909540118.

Hurst, M., & Cordes, S. (2018). A systematic investigation of the link between rational number processing and algebra ability. *British Journal of Psychology, 109*(1), 99–117. https://doi.org/10.1111/bjop.12244.

Jordan, N., Hansen, N., Fuchs, L., Siegler, R. S., Gersten, R., & Micklos, D. (2013). Developmental predictors of fraction concepts and procedures. *Journal of Experimental Child Psychology, 116*(1), 45–58. https://doi.org/10.1016/j.jecp.2013.02.001.

Kattou, M., Kontoyianni, K., Pitta-Pantazi, D., & Christou, C. (2013). Connecting mathematical creativity to mathematical ability. *ZDM - International Journal on Mathematics Education, 45*, 167–181. https://doi.org/10.1007/s11858-012-0467-1.

Lehtinen, E., Hannula-Sormunen, M. M., McMullen, J., & Gruber, H. (2017). Cultivating mathematical skills: From drill-and-practice to deliberate practice. *ZDM - Mathematics Education, 49*(4), https://doi.org/10.1007/s11858-017-0856-6.

Lortie-Forgues, H., Tian, J., & Siegler, R. S. (2015). Why is learning fraction and decimal arithmetic so difficult? *Developmental Review, 38*, 201–221. https://doi.org/10.1016/j.dr.2015.07.008.

Markovits, Z., & Sowder, J. (1994). Developing number Sense : An intervention study in grade 7. *Journal for Research in Mathematics Education, 25*(1), 4–29.

McMullen, J., Brezovszky, B., Hannula-Sormunen, M. M., Veermans, K., Rodríguez-Aflecht, G., Pongsakdi, N., et al. (2017). Adaptive number knowledge and its relation to arithmetic and pre-algebra knowledge. *Learning and Instruction, 49*. https://doi.org/10.1016/j.learninstruc.2017.02.001.

McMullen, J., Brezovszky, B., Rodríguez-Aflecht, G., Pongsakdi, N., Hannula-Sormunen, M. M., & Lehtinen, E. (2016). Adaptive number knowledge: Exploring the foundations of adaptivity with whole-number arithmetic. *Learning and Individual Differences, 47*, 172–181. https://doi.org/10.1016/j.lindif.2016.02.007.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational Numbers : Curriculum A new model and an experimental. *Journal for Research in Mathematics Education, 30*(2), 122–147. https://doi.org/10.2307/749607.

Mullis, I. V. S., Martin, M. O., Goh, S., & Cotter, K. (2016). *TIMSS 2015 encyclopedia: Education policy and curriculum in mathematics and science.* Retrieved from http://timssandpirls.bc.edu/timss2015/encyclopedia/ on June 2018 .

Muthén, L. K., & Muthén, B. O. (1998-2017). *Mplus user's guide* (7th ed.). Los Angeles, CA: Muthén and Muthén.

Nylund, K. L., Asparouhov, T., & Muthén, B. O. (2007). Deciding on the number of classes in latent class Analysis and growth mixture modeling: A Monte Carlo simulation study. *Structural Equation Modeling: A Multidisciplinary Journal, 14*(4), 535–569. https://doi.org/10.1080/10705510701575396.

Rittle-Johnson, B., & Siegler, R. S. (1998). The relation between conceptual and procedural knowledge in learning mathematics: A review. In C. Donlan (Ed.). *The development of mathematical skills* (pp. 75–110). East Sussex, UK: Psychology Press.

Schneider, M., Rittle-Johnson, B., & Star, J. R. (2011). Relations among conceptual knowledge, procedural knowledge, and procedural flexibility in two samples differing in prior knowledge. *Developmental Psychology, 47*(6), 1525–1538. https://doi.org/10.1037/a0024997.

Schneider, M., & Siegler, R. S. (2010). Representations of the magnitudes of fractions. *Journal of Experimental Psychology: Human Perception and Performance, 36*, 1227–1238. https://doi.org/10.1037/a0018170.

Schneider, M., & Stern, E. (2009). The inverse relation of addition and Subtraction : A knowledge integration perspective. *Mathematical Thinking and Learning, 11*, 92–101. https://doi.org/10.1080/10986060802584012.

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., et al. (2012). Early predictors of high school mathematics achievement. *Psychological Science, 23*, 691–697. https://doi.org/10.1177/0956797612440101.

Siegler, R. S., Thompson, C. A., & Opfer, J. E. (2009). The logarithmic-to-linear shift: One learning sequence, many tasks, many time scales. *Mind, Brain, and Education, 3*(3), 143–150. https://doi.org/10.1111/j.1751-228X.2009.01064.x.

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction, 14*(5), 503–518. https://doi.org/10.1016/j.learninstruc.2004.06.015.

Star, J. R. (2007). Foregrounding procedural knowledge. *Journal for Research in Mathematics Education, 38*(2), 132–135.

Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge.* New York: Springerhttps://doi.org/10.1007/978-1-4419-0591-8.

Torbeyns, J., Verschaffel, L., & Ghesquière, P. (2006). The development of children's adaptive expertise in the number domain 20 to 100. *Cognition and Instruction,* 439–465. https://doi.org/10.1207/s1532690xci2404.

Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2012). Naturally biased? In search for reaction time evidence for a natural number bias in adults. *The Journal of Mathematical Behavior, 31*(3), 344–355. https://doi.org/10.1016/j.jmathb.2012.02.

001.

Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students ' understanding of rational numbers and their notation. *Cognition and Instruction, 28*, 181–209. https://doi.org/10.1080/07370001003676603.

Van Hoof, J., Degrande, T., Ceulemans, E., Verschaffel, L., & Van Dooren, W. (2018). Towards a mathematically more correct understanding of rational numbers: A longitudinal study with upper elementary school learners. *Learning and Individual Differences, 61*, 99–108. https://doi.org/10.1016/j.lindif.2017.11.010.

Van Hoof, J., Janssen, R., Verschaffel, L., & Van Dooren, W. (2015). Inhibiting natural number knowledge in fourth graders: Towards a comprehensive test instrument. *ZDM - Mathematics Education, 47*(5), https://doi.org/10.1007/s11858-014-0650-7.

Verschaffel, L., Luwel, K., Torbeyns, J., & Van Dooren, W. (2009). Conceptualizing, investigating, and enhancing adaptive expertise in elementary mathematics education. *European Journal of Psychology of Education, 24*(3), 335–359. https://doi.org/10.1007/BF03174765.