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Putting Fractions Together

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Learning fractions is a critical step in children's mathematical development. However, many children struggle with learning fractions, especially fraction arithmetic. In this article, we propose a general framework for integrating understanding of individual fractions and fraction arithmetic, and we use the framework to generate interventions intended to improve understanding of both individual fractions and fraction addition. The framework, Putting Fractions Together (PFT), emphasizes that both individual fractions and sums of fractions are composed of unit fractions and can be represented by concatenating them (putting them together). To illustrate, both "3/9" and "2/9 + 1/9" can be represented by concatenating three 1/9s; similarly, 2/9 + 1/8 can be represented by concatenating two 1/9s and one 1/8. Interventions based on the PFT framework were tested in 2 experiments with fourth, fifth, and sixth grade children. The interventions led to improved performance on number line estimation and magnitude comparison tasks involving individual fractions and sums of fractions with equal and unequal denominators. Especially large improvements were observed on relatively difficult unequal-denominator fraction sum problems. The findings suggest that viewing individual nonunit fractions and sums of fractions as concatenations of unit fractions provides a sound conceptual foundation for improving children's knowledge of both. We discuss implications of the research for teaching and learning fractions, children's numerical development, and mathematics education in general.

Educational Impact and Implications Statement

Fractions are a uniquely important part of the mathematics curriculum in primary school. However, many children struggle with fractions, leading to difficulties with algebra and other more advanced mathematics. We developed an approach to teaching about fractions that emphasizes using unit fractions (fractions with a numerator of 1, such as 1/3) and the number line to think about both individual fractions and fraction addition. After briefly playing an educational computer game based on this approach, children displayed large improvements in their understanding of fractions and fraction addition. Incorporating this new approach into existing math curricula has the potential to improve children's learning of fractions.

Keywords: numerical development, fractions, arithmetic, estimation, educational games

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Learning fractions is a critical step in mathematical development. Individual differences in children's knowledge of fractions predict algebra proficiency and overall math achievement concurrently and over periods of at least 1 to 5 years, even after controlling for potential confounding variables including IQ, whole number arithmetic proficiency, and family SES (Booth & Newton, 2012; Booth, Newton, & Twiss-Garrity, 2014; Siegler et al., 2012). Fractions are also important in the workplace: 68% of participants in a large, nationally representative sample of American adults reported using fractions and other rational numbers in their jobs (Handel, 2016).

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Commensurate with their importance, substantial time is devoted to fractions instruction over several years of school (Common Core State Standards Initiative [CCSSI], 2010). However, despite the importance of fractions and fraction arithmetic, and the time devoted to their study, many children struggle with them (Hecht & Vagi, 2012; Jordan, Resnick, Rodrigues, Hansen, & Dyson, 2017), with the difficulties often persisting into adulthood (Schneider & Siegler, 2010; Stigler, Givvin, & Thompson, 2010).

In the present study, we propose a conceptual framework designed to improve children's understanding of fractions and fraction arithmetic. The framework is called *Putting Fractions Together* (PFT), because it emphasizes that individual fractions and sums of fractions are both composed of unit fractions and can be represented by concatenating (putting together) unit fractions. We report results of two experiments investigating effects of playing an educational computer game based on the PFT framework. We first briefly review research on children's difficulties with fractions. Then we describe the PFT framework and introduce the present study.

Children's Difficulties With Fractions

Children have difficulty understanding both individual fractions and fraction arithmetic. A central obstacle to understanding individual fractions is the *whole number bias*, a tendency to think of a fraction as two separate whole numbers rather than as a single number (Mack, 1995; Meert, Grégoire, & Noël, 2009; Ni & Zhou, 2005). Whole number bias leads to errors involving fraction comparison, such as claiming that $2/9 > 1/2$ because $2 > 1$ and $9 > 2$ (Fazio, DeWolf, & Siegler, 2016; Mazzocco & Devlin, 2008). The bias also interferes with understanding fraction equivalence; for example, it leads children to represent 9/18 as larger than 1/2 on a number line (Braithwaite & Siegler, 2018; Bright, Behr, Post, & Wachsmuth, 1988). These and other difficulties understanding individual fractions lead to results such as 50% of U.S. eighth graders who took a national achievement test failing to correctly order 5/9, 2/7, and 1/2 (U.S. Department of Education, Institute of Education Sciences, 2007).

Whole number bias is also a major source of fraction arithmetic errors. For example, in Siegler and Pyke (2013), sixth and eighth graders erred on more than 20% of fraction arithmetic items by treating numerators and denominators as independent whole numbers, for example when claiming that $3/5 + 1/4 = 4/9$. These errors may also reflect overgeneralization of the procedure for multiplying fractions (Braithwaite, Pyke, & Siegler, 2017). Similar errors have been documented in numerous other studies of fraction arithmetic (Byrnes & Wasik, 1991; Gabriel et al., 2013; Hecht, 1998; Siegler, Thompson, & Schneider, 2011).

Beyond their difficulties calculating exact answers to fraction arithmetic problems, many children lack a sense of the approximate sizes of the answers (Hecht, 1998; Hecht & Vagi, 2012; Siegler & Lortie-Forgues, 2015). For example, on an early National Assessment of Educational Progress (NAEP), when asked to choose the best estimate of the answer to $12/13 + 7/8$ from the options 1, 2, 19, and 21, only 24% of U.S. 13-year-olds chose the correct answer (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980). More recently, when asked to estimate sums of pairs of fractions on a number line, U.S. seventh and eighth graders' estimates were no more accurate than if they had ignored the numbers involved and simply marked the midpoint of the line on every trial (Braithwaite, Tian, & Siegler, 2018). Strikingly, on a majority of trials in Braithwaite et al. (2018), middle school students' number line estimates of answers to fraction addition problems were smaller than their own estimates of one or both addends. These findings suggest that many children do not understand the meaning of even the most basic arithmetic operation, addition, in the context of fractions.

Inability to represent and reason about the magnitudes (i.e., sizes) of fractions seems to be at the heart of many difficulties with fractions and fraction arithmetic, including the whole number bias and confusing different fraction arithmetic operations (Fuchs et al., 2013; Hamdan & Gunderson, 2017; Hansen et al., 2015; Siegler et al., 2011). Claiming that $2/9 > 1/2$, representing 9/18 as larger than 1/2, and being unable to order correctly three fractions with single digit numerators and denominators all reflect inaccurate understanding of the magnitudes of the fractions involved. Similarly, claiming that $3/5 + 1/4 = 4/9$, despite 4/9 being less than 3/5, violates a basic principle connecting arithmetic to numerical magnitudes: a sum of positive numbers must be greater than any addend in the sum. In summary, poor understanding of fraction magnitudes is a common thread running through common errors on many tasks involving individual fractions and fraction arithmetic.

Putting Fractions Together

This analysis suggests a need to improve not only children's understanding of the magnitudes of individual fractions, but also their ability to reason about fraction magnitudes in the context of arithmetic. The PFT framework was created to achieve these goals. The central idea of the framework is that both individual fractions and sums of fractions are composed of unit fractions and therefore can be understood in the same way. Consistent with this idea, PFT specifies that individual fractions and fraction sums are represented by concatenating, or "putting together," representations of unit fractions.

PFT was motivated by noting several closely analogous features of whole numbers and fractions (see Table 1). As a whole number represents a quantity of ones, so a fraction represents a quantity of unit fractions; ones and unit fractions serve as units for whole numbers and fractions, respectively. For both whole numbers and fractions, a sum represents a number of units. Individual whole numbers and whole number sums can be generated by counting. For example, one may generate 3 by counting "1, 2, 3," and one may generate $3 + 2$ by counting "1, 2, 3" for the first addend and then "4, 5" for the second addend. Thus, counting connects individual whole numbers with whole number arithmetic.

Concatenating unit fractions plays an analogous role for individual fractions and fraction arithmetic. For example, 3/7 can be generated by concatenating three 1/7s; $3/7 + 2/7$ can be generated by concatenating three 1/7s and two 1/7s, and $3/7 + 2/5$ can be generated by concatenating three 1/7s and two 1/5s. Thus, the PFT approach was intended to connect individual fractions with fraction arithmetic in the same way that individual whole numbers are connected to whole number arithmetic.

Figure 1 shows a visual representation of this way of thinking about individual fractions and the relation between them and fraction sums. Unit fractions are represented by fraction strips with lengths inversely proportional to their denominators (Figure 1A).

Table 1
Analogy Between Whole Numbers and Fractions

Aspect of the analogy	Whole numbers	Fractions
The unit is . . .	One (“1”)	A unit fraction (“1/2”, “1/3”, “1/4”, etc.)
A number means . . .	A quantity of ones (“3” means “3 ones”)	A quantity of unit fractions (“3/7” means “3 1/7s”)
A sum means . . .	A combination of quantities of ones (“3 + 2” means “3 ones and 2 ones”)	A combination of quantities of unit fractions (“3/7 + 2/5” means “3 1/7s and 2 1/5s”)
The magnitude of a number or sum can be generated by . . .	Counting ones	Concatenating unit fractions

Nonunit fractions are represented by concatenating fraction strips whose unit is indicated by the fraction’s denominator and whose number of iterations is indicated by the fraction’s numerator (Figure 1B). Sums of fractions are represented by concatenating fraction strips that represent the addends; this procedure can be used for sums of fractions with equal denominators (Figure 1C) or unequal denominators (Figure 1D). Finally, magnitudes of fractions and fraction sums are represented as positions on a number line (Figures 1B, 1C, and 1D). The position of a fraction or fraction sum can be found by placing the appropriate unit fractions above the line beginning at 0; the right edge of the rightmost unit fraction indicates the fraction’s or fraction sum’s magnitude, that is, its position on the number line.

Several aspects of PFT have been proposed previously. PFT’s emphasis on conceptualizing individual fractions as concatenations of unit fractions is shared with existing approaches to fraction instruction. For example, the Common Core State Standards for Mathematics recommends that students “understand a fraction *a*/*b* with $a > 1$ as a sum of fractions 1/*b*” (4.NF.B.3; CCSSI, 2010), an approach consistent with that of PFT in that both conceptualize a fraction as a quantity of unit fractions. The Standards also advocate that students “represent a fraction *a*/*b* on a number line diagram by marking off *a* lengths 1/*b* from 0” (3.NF.A.2.B; CCSSI, 2010), a recommendation consistent with the implementations of PFT shown in Figure 1.

There is substantial theoretical and empirical basis for inclusion of these ideas in the Common Core Standards. For example, Tzur (1999) described the approach of constructing fractions by iterating unit fractions, and Steffe (2001) noted the analogy between iterating unit fractions and counting with whole numbers (see also Steffe, 2004; Tzur & Hunt, 2015). Children’s understanding of these concepts begins with iterating unit fractions to make up a whole, continues with construction of proper fractions, and then is extended to improper fractions (Norton & Wilkins, 2009; Wilkins & Norton, 2018).

Although previous work has articulated the role of iterating unit fractions in children’s understanding of individual fractions, it has not emphasized the utility of these ideas for understanding fraction addition. A key innovation of PFT is the use of a single procedure, concatenating unit fractions, to represent not only individual fractions but also fraction sums with both equal and unequal denominators (see Wu, 2009, for a similar proposal). In terms similar to those of the Common Core Standards, PFT advocates that students “understand a sum of fractions *a*/*b* + *c*/*d* as a sum of fractions 1/*b* and 1/*d*” and “represent a sum of fractions *a*/*b* + *c*/*d* on a number line diagram by first marking off *a* lengths 1/*b* from 0, then marking off *c* lengths 1/*d* from *a*/*b*.” This approach to fraction addition is not mentioned in the Common Core Standards, and we believe that it is an important contribution of PFT.

Learning to view fraction addition in this way could help children to avoid common errors. As noted earlier, many children add fractions by adding their numerators and denominators, as in 3/5 + 1/4 = 4/9 (Siegler & Pyke, 2013). From the perspective of PFT, 3/5 + 1/4 and 4/9 each comprise the same number of unit fractions (i.e., four). However, all unit fractions comprising 3/5 + 1/4 (i.e., three 1/5ths and one 1/4th) are larger than any of the unit fractions comprising 4/9 (i.e., four 1/9ths). Thus, 3/5 + 1/4 cannot equal 4/9; 3/5 + 1/4 must be larger. This reasoning could help children understand why it makes no sense to add fractions by adding their numerators and denominators. Similarly, as noted earlier, children often estimate a sum of two positive fractions to be smaller than one of the addends; this occurred on 52% of trials in Braithwaite et al. (2018). PFT could help children to avoid such errors by illustrating how a sum of positive fractions contains each addend and therefore must exceed the individual addends.

PFT may also offer several more general advantages for learning about fractions. First, by emphasizing that both individual fractions and fraction sums are composed of unit fractions, PFT could help children to connect their understanding of individual fractions and fraction addition. Second, by highlighting aspects of fractions that are analogous to aspects of whole numbers, PFT could leverage children’s whole number knowledge to help, rather than hinder, learning about fractions. Third, PFT offers a way to understand what it means to add two fractions that does not depend on understanding the procedures required to calculate fraction sums. Related, PFT provides a method for estimating the approximate sizes of fraction sums without calculating the exact answers.

The Present Study

Although some aspects of the PFT framework have been described previously (CCSSI, 2010; Steffe, 2001; Tzur, 1999), to our knowledge, the effectiveness of this approach for improving understanding of either individual fractions or fraction arithmetic has not been empirically tested. Consistent with this conclusion, in a recent comprehensive review of fraction interventions aimed at struggling math learners (Roesslein & Codding, 2018), no intervention was cited that included the main elements of PFT.

The present study examined whether interventions based on the PFT framework could improve children’s knowledge of the magnitudes of individual fractions and fraction sums. The interventions built on a computer game developed by Fazio, Kennedy, and

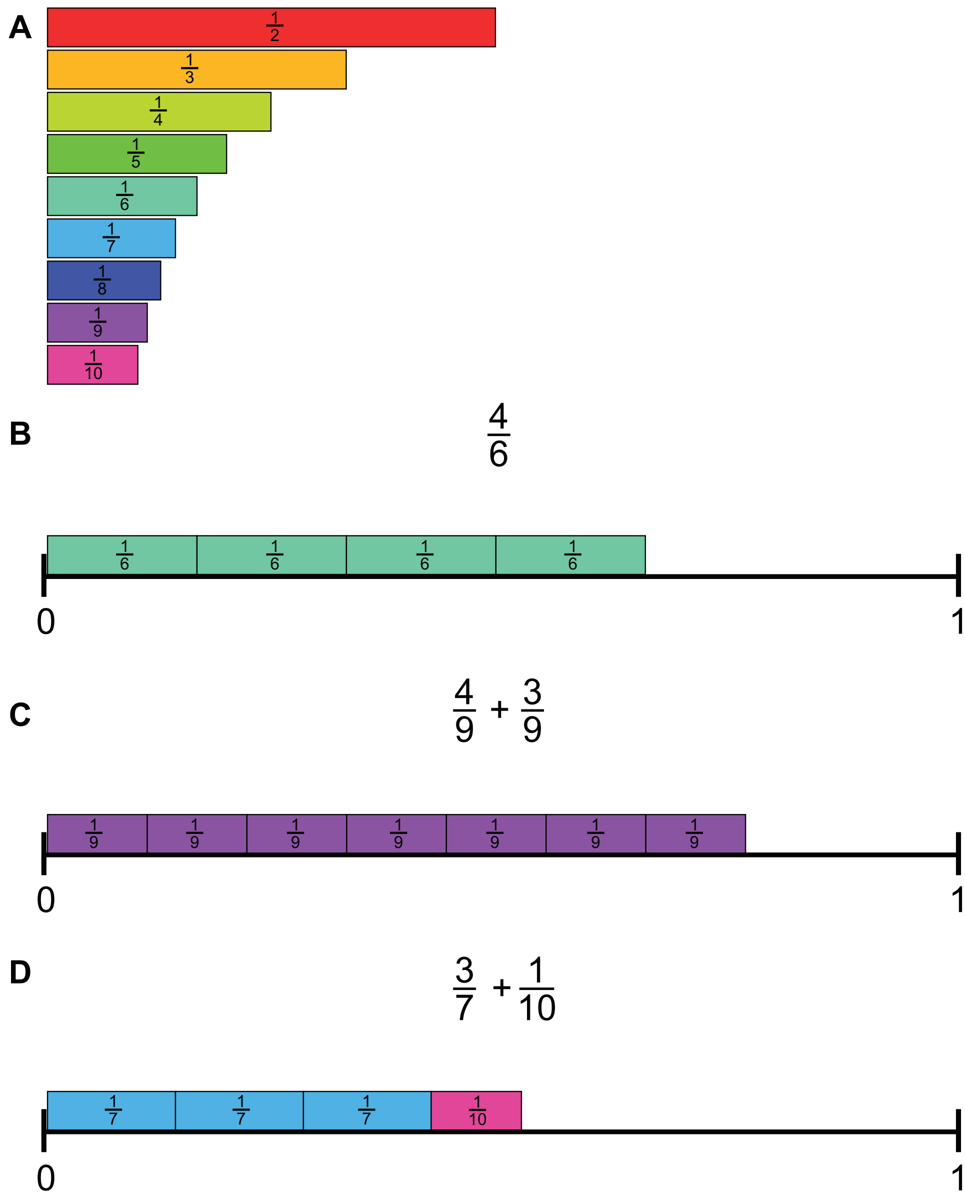

Figure 1. Visual representation of the key components of Putting Fractions Together (PFT). A: Fraction strips. B: Representation of an individual fraction. C: Representation of an equal-denominator fraction sum. D: Representation of an unequal-denominator fraction sum. See the online article for the color version of this figure.

Siegler (2016). In that earlier intervention, fourth and fifth graders first were told how to locate a fraction on a number line by partitioning the line into segments and counting the appropriate number of segments. Then the children were presented fractions, told that each fraction indicated a hidden monster's location on a number line, and asked to use the fraction to estimate the monster's location. Children "caught the monster" if their estimate was sufficiently close to the correct location; the monster "escaped" if the estimate was not close enough. The intervention led to gains from pretest to posttest in the accuracy of fraction number line estimates and in percent correct on a fraction magnitude comparison task.

Understanding of fraction arithmetic was not assessed by Fazio et al. (2016), but an initial test of that intervention that we conducted indicated no improvement from pretest to posttest in estimation of unequal-denominator fraction sums. The reason for the lack of improvement with unequal-denominator fraction sums may relate to the strategy for number line estimation emphasized within the earlier intervention (and by many mathematics textbooks)—partitioning the number line into the number of segments indicated by the denominator and then counting the number of segments indicated by the numerator. In our initial testing, we observed that many students could use this strategy competently to estimate individual fractions and equal-denominator sums but could not use it to estimate unequal-denominator sums. For example, when asked to estimate 1/3 + 1/2, a student first partitioned the line into thirds to estimate 1/3, but then did not know how to add on the 1/2.

Thus, instruction in partitioning may be a useful way to help students understand the magnitudes of individual fractions and addition of fractions with equal denominators, but it might not help them understand addition with unequal denominators.

The PFT framework suggested a more generally applicable approach in which fractions are represented by concatenating unit fractions that are separate from the number line itself. This approach allows representation not only of individual fractions and of sums of fractions with equal denominators but also of sums of fractions with unequal denominators. We predicted that instruction based on this approach would improve children's understanding of both individual fractions and sums of fractions with both equal and unequal denominators.

To test these predictions, we presented children a game (see Figure 2) that involved concatenating fraction strips rather than partitioning the number line into segments. While playing the game, children needed to choose among fraction strips and move them just above the number line to generate the answers. Fraction strips also appeared as part of the feedback to children's answers.

We created three interventions that differed in the targets that children practiced estimating. In the *individual-fractions intervention*, children estimated single fractions (Figure 2A). In the

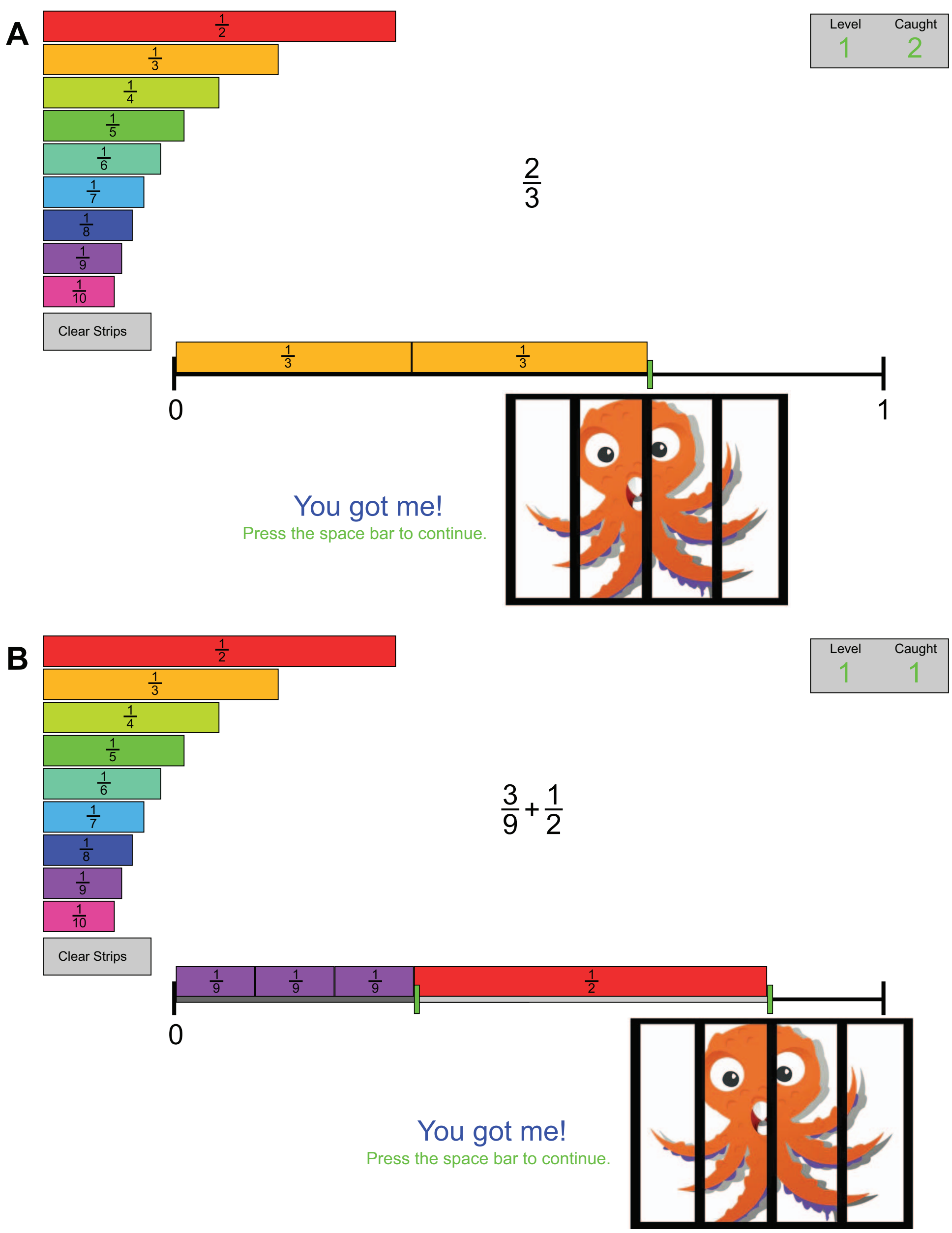

Figure 2. Example trials from the computer game used in the fraction interventions tested in Experiments 1 and 2, involving (A) an individual fraction, and (B) a fraction sum. Each example displays the feedback children received after "catching the monster." See the online article for the color version of this figure.

fraction-sums intervention, children estimated fraction sums (Figure 2B). In the *individual-fractions-and-sums intervention*, children estimated both individual fractions and fraction sums. Experiment 1 tested the individual-fractions and individual-fractions-and-sums interventions; Experiment 2 tested the fraction-sums intervention and an active control intervention involving whole number sums. The specific questions addressed by the interventions are presented in the introductions to the two experiments.

Experiment 1

Experiment 1 was conducted to answer two questions. First, would interventions based on PFT improve children's understanding of fraction addition with both equal and unequal denominators, as well as their understanding of individual fractions? Second, if so, which parts of the PFT framework are necessary to achieve the improvements in understanding of fraction addition? Representing individual fractions by concatenating unit fractions could give children the insight that fraction sums can also be represented in this way, resulting in improved understanding of fraction addition even without fraction addition instruction; this would seem especially likely with sums of equal-denominator fractions. On the other hand, instruction and practice in PFT-based procedures for representing fraction sums might be required to achieve this improvement. To address these questions, children were randomly assigned to either the individual-fractions intervention or the individual-fractions-and-sums intervention.

Four assessment tasks were administered before and after the interventions. Two tasks, fraction number line estimation and fraction magnitude comparison, assessed understanding of individual fractions. The other two, number line estimation with equal- and unequal-denominator sums, assessed understanding of fraction addition. In all tasks, performance after the interventions was assessed using stimuli (individual fractions or fraction sums) that were not presented during the intervention. Thus, any improvement would reflect generalization of training to novel items rather than memorization of practiced items.

We predicted that performance on the tasks assessing understanding of individual fractions would improve following both interventions. Amount of improvement in knowledge of individual fractions was not expected to differ between conditions, because both interventions included the same instruction and practice with individual fractions. We also predicted improvement on the tasks assessing understanding of fraction addition, with the improvement expected to be greater after the individual-fractions-and-sums intervention for the unequal denominator sum problems. Strategies for estimating individual fractions may generalize easily to estimating equal-denominator sums, because both estimates can be generated by concatenating unit fractions with the same denominators. However, concatenating unit fractions with unequal denominators seemed likely to require instruction and practice in how that can be done.

Participants were fourth- and fifth-grade children. The reason was that individual fractions and fraction addition with equal denominators are a major focus of mathematics education in fourth grade, and fraction addition with unequal denominators is typically taught in fifth grade (CCSSI, 2010).

Method

Participants. The participants, 63 fourth and fifth graders (9- to 11-year-olds), were randomly assigned either to the individual-fractions-and-sums condition (N = 32, 15 fourth graders and 17 fifth graders; 10 boys and 20 girls [sex was not recorded for two children due to experimenter error]) or to the individual-fractions condition (N = 31, 15 fourth graders and 16 fifth graders, 14 boys and 17 girls). All children attended a middle school in the Pittsburgh, Pennsylvania area at which 39% of students were eligible for free or reduced-price lunch, and at which 95% of students were Caucasian, 2% African American, and 3% Other. Experimenters were David W. Braithwaite, two female research assistants, and a male postdoctoral research associate. The experiment was conducted with the approval of the Carnegie Mellon University Institutional Review Board (Study #00000396).

General procedure. The study included two sessions. In the first, children completed the pretest, Part 1 of the intervention, and the midtest.[1] In the second, children completed Part 2 of the intervention and the posttest. The two sessions were conducted on successive days for 41 children and two to six days apart for the other 22. No differences in performance were found between children who did or did not receive the sessions on successive days on any task at any time of measurement. The interventions and assessments were administered on a computer by an experimenter working with a child one-on-one.

Interventions. Each of the two parts of the intervention consisted of a tutorial followed by gameplay. Part 1 of the intervention took an average of 20.5 min in both conditions; Part 2 averaged 19.0 min in the individual-fractions-and-sums condition; and 17.6 min in the individual-fractions condition. Time playing the game in each part of the intervention was the same in both conditions (see below); the 1.4-min difference between conditions in the length of Part 2 therefore reflected a difference in tutorial time. The scripts that experimenters followed for giving instructions and feedback during the intervention are provided in the Supplement 1 of the online supplementary materials.

Part 1. This part of the intervention was identical for children in the two conditions. Children first received a tutorial that introduced them to the fraction strips representing unit fractions from 1/2 to 1/10; then, the experimenter explained a procedure for using fraction strips to position fractions on the number line. The procedure involved selecting the appropriate type and number of fraction strips and dragging them just above the number line to create a graphical representation of the fraction (Figure 1B). Children used this procedure to place five fractions on the line. If a child used the fraction strips incorrectly, for example by choosing fraction strips matching the target fraction's numerator instead of its denominator, the experimenter pointed out the error and guided the child to perform the correct procedure.

Children then played the game. On each trial, a fraction appeared above a 0–1 number line. The task was to show the size of the fraction by clicking at its location on the number line. If children clicked near the correct location, with the criterion for

[1] For six children, the first session was conducted over 2 days, the pretest on the first day and Part 1 of the intervention and the midtest on the second day. These children's second session, including Part 2 of the intervention and the posttest, was conducted on a later day, as with the other children.

"near" changing as the children gained skill in playing the game, the monster was "caught" inside a cage (Figure 2A); otherwise, the monster "escaped" and chuckled wickedly.

Gameplay in both conditions progressed through three phases, each lasting 4.5 min. Children completed as many items as they could during that time (an average of 66.7 trials in the individual-fractions condition and 74.2 trials in the individual-fractions-and-sums condition). In Phase 1, children could, and almost all children did, select fraction strips and move them onto the number line, using the procedure practiced during the tutorial. If a child used the fraction strips incorrectly, resulting in the monster escaping, the experimenter explained the error before the next trial. In Phase 2, the strips became immovable, and children were encouraged to imagine moving them onto the number line. In Phase 3, the strips were hidden, so children had to answer without moving or seeing them. This "concreteness fading" approach (Fyfe, McNeil, Son, & Goldstone, 2014) was intended to encourage children to transition from relying on perception and manipulation of the fraction strips to relying on mental representations. In all three phases, after children responded on each trial, fraction strips representing the target fraction appeared above the number line; in Phase 1, fraction strips placed by children disappeared before the correct fraction strips appeared.

Each phase was subdivided into three levels, in which increasingly accurate responses were needed to catch the monster. For the monster to be caught, the distance between a child's response and the fraction's location needed to be within 20% of the number line's length at Level 1, 15% at Level 2, and 10% at Level 3. If children caught the monster four consecutive times, they progressed to the next level, unless they were already at the highest level. Each new level involved a smaller monster than the previous one, to lend plausibility to the need for increasingly accurate estimates.

The fractions that were presented had values approximately evenly distributed from 0 to 1, with denominators from 2 to 10 appearing equally often. Fractions that later appeared on the number line estimation task on the midtest did not appear during the tutorial or game.

Part 2: Individual-fractions-and-sums condition. Children in the individual-fractions-and-sums condition received a tutorial introducing a procedure for using fraction strips to find the location of a sum of two fractions on the number line. This procedure involved concatenating fraction strips representing the two addends (Figure 1D). Children used the procedure to place three unequal-denominator fraction sums on the number line. As in Part 1, If a child used the fraction strips incorrectly, the experimenter guided the child to perform the correct procedure.

Children then played the fraction sums game (Figure 2B). They were asked to click on the line twice for each trial—once to show the size of the first addend and then to show the size of the sum. The monster appeared after the second click, the location of which determined whether the monster was caught. After children responded, fraction strips representing the target sum appeared on the number line; during Phase 1, fraction strips placed by children disappeared before the correct fraction strips appeared. Gameplay progressed as in Part 1, involving the same three phases of 4.5 min each and the same three levels within each phase. During Phase 1, if a child used the fraction strips incorrectly, feedback was given as in the tutorial. Children completed an average of 49.2 trials.

All sums involved unequal-denominator addends and had values less than 1; sums were below 1/2 on 46% of trials and above 1/2 on 54% of trials. Denominators between 2 and 10 appeared equally often. Sums that appeared in the number line estimation task on the posttest did not appear during the intervention, nor did sums involving the same addends in reversed order.

Part 2: Individual-fractions condition. In the second part of the intervention, children in the individual-fractions condition received a review of the fraction strips and the procedure for estimating individual fractions. They used this procedure to place three fractions on the number line. They then played the game with individual fractions, as in Part 1, which meant that they spent twice as much time playing the individual fractions game as did children in the individual-fractions-and-sums condition. Feedback was given as in Part 1. Children completed an average of 82.3 trials.

Assessments. The pretest included two individual fractions tasks—number line estimation with individual fractions and fraction magnitude comparison—and two tasks involving fraction sums—number line estimation with equal-denominator sums and number line estimation with unequal-denominator sums. The midtest consisted of the two tasks involving individual fractions; the posttest consisted of the two tasks involving fraction sums. In this and the next experiment, two sets of stimuli were used for each task, one for the pretest and the other for the midtest or posttest. Which set of stimuli was used for the pretest was counterbalanced within each condition. In both experiments, stimuli for each task were presented to each participant in a different random order. All assessment items are provided in the Supplement 1 of the online supplementary materials.

Number line estimation of individual fractions. On each trial, a fraction appeared above the center of a 0–1 number line. The task was to mark the location of the fraction on the number line. Stimuli were two sets of 12 fractions, 3 in each quartile from 0 to 1. Children did not have fraction strips available on this or any other pretest or posttest task.

Fraction magnitude comparison. Children were shown a 0–1 number line, with 3/5 marked on the line. On each trial, a different fraction appeared under the center of the number line, and children were asked whether the fraction was less than or greater than 3/5. Each set of comparison items contained 15 fractions with denominators from 3 to 10, 8 smaller than 3/5 and 7 larger than 3/5.

Number line estimation of equal-denominator fraction sums. Children were shown an addition problem involving fractions with equal denominators (e.g., 3/8 + 2/8) above a 0–1 number line and instructed to click on the line to mark the location of the first addend and then click again to mark the location of the sum. These instructions were intended to encourage children to create a visual reference point for the location of the first addend before estimating the sum. Only the last mark for each trial affected children's scores. Stimuli were two sets of 8 equal-denominator fraction sums, four in each set with answers less than or equal to 1/2 and four with answers between 1/2 and 1. For five of the eight sums in each set, at least one of the addends was a unit fraction. Each number from 3 to 10 appeared as the denominator of the addends on one item in each set.

Number line estimation of unequal-denominator fraction sums. This task was the same as the previous one, except that the addition problems involved fractions with unequal denominators (e.g., 3/7 + 1/10). Stimuli were two sets of 9 problems, four with sums less than or equal to 1/2 and five with sums greater than 1/2 but less than 1. For all but two of the sums (both in the same set), at least one of the addends was a unit fraction. Each denominator from 2 to 10 appeared twice in each set.

Analyses. Performance on the number line estimation tasks was measured using percent absolute error (PAE), defined as |Participant's Answer – Correct Answer|/Numerical Range × 100. For example, if a participant was asked to estimate 1/2 + 1/4 and marked the location corresponding to 0.65, the PAE for that trial would be |0.65 – 0.75|/1 × 100 = 10. Lower PAE indicates higher accuracy. Dividing by numerical range (which was always 1 in the present study) permits meaningful comparison between PAEs derived from tasks that use different numerical ranges, while ensuring that the same distance between the child's estimate and the correct location on the number line always translates to the same PAE within a task. The measure of magnitude comparison accuracy was percent correct choices.

For each measure, change scores (i.e., difference between pretest and midtest or posttest) were submitted to analyses of covariance (ANCOVA) with condition and grade as between-subjects factors and pretest score as a covariate. All significant effects are reported. All reported effects remained if the covariate was excluded from the analysis, if the test version used on the pretest was included as a factor, or both. All reported effects of condition also appeared as interactions of condition with time of test when the data were analyzed using performance as the dependent variable and time of test as a within-subjects factor.

Next, to determine whether performance improved following the intervention, paired t tests were conducted to compare pretest scores to midtest or posttest scores. If change scores differed by condition in the ANCOVA, separate t tests were conducted for the two conditions; if not, a single t test was conducted, combining across conditions. Estimates of effect size (d) were calculated using the formula $d = t * [2 * (1 - r)/N]^{1/2}$, where t is the value obtained from the t test, r is the pretest-posttest correlation, and N is the sample size; this formula corrects for correlations between repeated measures (Dunlap, Cortina, Vaslow, & Burke, 1996).

Results

Table 2 shows mean pretest and midtest or posttest performance, change in performance, and estimated marginal mean change on all tasks within each condition. (These data are presented separated by grade in the Supplement 3 of the online supplementary materials).

Number line estimation of individual fractions. As expected, because the treatments were identical from pretest to midtest, changes in PAE from pretest to midtest did not differ between conditions, $F(1, 59) < .001$, $p = .99$, $\eta_g^2 < .001$. Across conditions, PAE improved from 11.7 to 6.6, $t(62) = 5.3$, $p < .001$, $d = 0.62$.

Fraction magnitude comparison. As expected for the same reason, change in accuracy from pretest to midtest also did not differ between conditions, $F(1, 59) = 1.8$, $p = .18$, $\eta_g^2 = .030$. Accuracy increased marginally across conditions, from 75.0% to 78.9%, $t(62) = 1.7$, $p = .10$, $d = 0.20$.

Number line estimation of equal-denominator sums. On equal-denominator sums, change in PAE from pretest to posttest did not differ between conditions, $F(1, 59) = 0.42$, $p = .52$, $\eta_g^2 =$ .007. Across conditions, PAE improved from 16.5 to 9.3, $t(62) = 6.5$, $p < .001$, $d = 0.75$.

Table 2
Mean (Standard Deviation) Pretest and Midtest or Posttest Performance and Change in Performance by Condition (Experiment 1)

Test measure	Individual-fractions-and-sums condition	Individual-fractions condition
Number line estimation of individual fractions – PAE		
Pretest	10.8 (8.7)	12.6 (9.8)
Midtest	6.4 (4.2)	6.9 (5.0)
Change	−4.5 (7.0)	−5.7 (8.3)
EMM of change	−5.0	−5.0
Fraction magnitude comparison – % correct		
Pretest	77.3 (19.9)	72.7 (18.5)
Midtest	83.1 (17.4)	74.6 (22.5)
Change	5.8 (13.5)	1.9 (22.8)
EMM of change	6.5	0.9
Number line estimation of equal-denominator sums – PAE		
Pretest	16.6 (9.4)	16.5 (11.6)
Posttest	8.8 (6.5)	9.9 (9.4)
Change	−7.7 (9.5)	−6.6 (8.1)
EMM of change	−7.7	−6.6
Number line estimation of unequal-denominator sums – PAE		
Pretest	22.4 (11.3)	22.4 (10.7)
Posttest	8.3 (5.4)	13.8 (8.9)
Change	−14.2 (10.8)	−8.6 (7.9)
EMM of change	−14.1	−8.5

Note. EMM = estimated marginal mean (EMMs were derived from the analyses of covariance described in the main text and are adjusted for the covariate [pretest]); PAE = percent absolute error.

Number line estimation of unequal-denominator sums. On unequal-denominator sums, PAE improved by considerably more from pretest to posttest in the individual-fractions-and-sums condition (pretest: 22.4, posttest: 8.3, change: 14.2) than in the individual-fractions condition (pretest: 22.4, posttest: 13.8, change: 8.6), $F(1, 59) = 12.4$, $p < .001$, $\eta_g^2 = .17$. The improvements in both conditions were significant, $t(31) = 7.4$, $p < .001$, $d = 1.52$ for the individual-fractions-and-sums condition and $t(30) = 6.0$, $p < .001$, $d = 0.85$ for the individual-fractions condition.

Discussion

Experiment 1 replicated the improvements in children's number line estimates of individual fractions found by Fazio et al. (2016). Improved accuracy on the fraction magnitude comparison task was replicated marginally.

Children's estimates of equal- and unequal-denominator fraction sums improved substantially in both conditions (d ranging from 0.75 to 1.52). These large improvements reflected transfer of learning to novel addition problems that were not shown during the intervention. Thus, instruction and practice based on the PFT framework improved children's understanding of fraction addition.

Although estimation accuracy for unequal-denominator sums improved in both conditions, the gains were larger in the individual-fractions-and-sums condition than in the individual-fractions condition. These findings suggest two lessons. First, modeling the structure of individual fractions by concatenating unit fractions helps to improve children's understanding of fraction addition even without explicitly modeling fraction addition. Second, explicitly modeling addition of fractions with unequal denominators appears necessary for children to enjoy the full benefits of PFT for understanding unequal-denominator addition.

Experiment 2

In Experiment 2, we tested whether an intervention based on PFT would improve children's performance on a transfer task that involved comparison of fraction sums to one. Reasoning about fractions larger than one represents a major conceptual advance in children's fraction learning trajectories (Norton & Wilkins, 2009; Wilkins & Norton, 2018), so it would be noteworthy if children's reasoning about fraction sums larger than one improved after the intervention despite no such sums being presented during the intervention. Children also did not compare fraction sums to any specific number during the intervention. This skill seems valuable, because it could help children to recognize the many instances when common arithmetic errors yield implausible answers. For example, the skill could have helped the 23% of children in Siegler and Pyke (2013) who claimed that 2/3 + 3/5 = 5/8 to recognize that 2/3 + 3/5 is greater than one, that 5/8 is not greater than one, and therefore that 5/8 could not be the correct answer. This recognition might motivate such children to try a different addition strategy. Siegler and Pyke (2013) found that the same children who used flawed arithmetic procedures on one problem of a given type (e.g., addition with unequal denominators) often used a correct procedure on another problem of the same type. Thus, if children recognize that an answer is implausible, they often would know a correct procedure and might well try it.

The transfer task allowed us to distinguish between two interpretations of the results of Experiment 1. We interpreted the Experiment 1 findings as indicating that PFT-based interventions improved children's understanding of fraction addition. An alternative interpretation, however, is that children merely learned a task-specific procedure for estimating sums on a number line. The former interpretation, but not the latter, suggests that PFT-based interventions should lead to improved accuracy on the transfer task.

To test whether improved accuracy on the transfer task (if observed) resulted from increased use of estimation strategies, we administered a version of the transfer task in which children provided concurrent strategy reports. We were particularly interested in children's estimation strategies, because we hypothesized that the effectiveness of the fraction-sums intervention was due to it leading children to use estimation strategies more frequently. Thus, we analyzed whether children's use of estimation strategies increased after the intervention and whether changes in individual children's use of estimation strategies were related to changes in their accuracy.

Participants in Experiment 2 were randomly assigned to one of two interventions. The fraction-sums intervention was an abbreviated version of the individual-fractions-and-sums intervention of Experiment 1. In the new intervention, children only estimated fraction sums. Testing this intervention enabled us to determine whether children could benefit from the fraction sum game without prior practice estimating individual fraction magnitudes. It seemed plausible that they could, because accurately estimating fraction sums requires accurately estimating the individual fraction addends.

The whole-number-sums intervention served as an active control condition. It involved estimating whole number sums on a 0–1,000 number line, instead of fraction sums on a 0–1 number line, and using whole number strips analogous to the fraction strips in the fraction-sums condition. Thus, the whole-number-sums intervention controlled for experience with the experimenter, the experimental situation, the procedure of using physical and imagined parts to generate a larger sum, and number line estimation.

Before and after the interventions, children completed the transfer tasks and number line estimation with unequal-denominator fraction sums. To enable completion of all tasks within a single session, tasks involving individual fractions were not included, nor was the number line estimation task with equal-denominator fraction sums. Participants were fifth and sixth grade children, because fraction addition is a major focus of mathematics education in fifth and sixth grades (CCSSI, 2010).

We predicted that (a) the fraction-sums intervention would lead to improved performance on the number line estimation task with unequal-denominator fraction sums; (b) the fraction-sums intervention would also lead to improved performance on the transfer tasks; (c) these improvements would be accompanied by increased use of estimation strategies; and (d) on all assessment tasks, children who received the whole-number-sums intervention would show either no improvement or smaller improvement than those who received the fraction-sums intervention.

Method

Participants. Participants were 104 fifth and sixth graders. The fraction-sums condition included 53 children (23 fifth graders and 30 sixth graders; 17 boys and 35 girls; one child did not report gender). The whole-number-sums condition included 51 children (22 fifth graders and 29 sixth graders; 26 boys and 25 girls). Children's ages ranged from 10 to 12 years. Fourteen sixth graders (seven in each condition) were recruited from a middle school in Pittsburgh, Pennsylvania in which 84% of students were eligible for free or reduced-price lunch and whose students were 68% Caucasian, 23% African American, 8% biracial, and 1% Hispanic or Latino. The remaining 90 children were recruited from a school in Tallahassee, Florida in which 28% of students were eligible for free or reduced-price lunch and in which 52% of students were Caucasian, 28% African American, 19% Hispanic or Latino, and 2% Other. The experimenters were David W. Braithwaite and three research assistants, one male and two females. Four children were excluded from some or all analyses because they did not finish the experiment or because of experimenter error; details are provided in Supplement 5 in the online supplementary materials. Data collection was conducted in Pittsburgh under the same institutional review board (IRB) approval as Experiment 1, and in Tallahassee with the approval of the Florida State University IRB (Study #29739).

General procedure. Each child completed the pretest, intervention, and posttest during a single session. The intervention and assessments were administered on a laptop computer by an experimenter working with children one-on-one.

Interventions. As in Experiment 1, the intervention consisted of a tutorial followed by gameplay. The intervention took an average of 21.1 min in the fraction-sums condition and 21.3 min in the whole-number-sums condition. The scripts that experimenters followed for giving instructions and feedback are provided in the Supplement 2 in the online supplementary materials.

Fraction-sums condition. Children were first introduced to the fraction strips and the procedure for using them to find the location of a fraction on the number line, as in Part 1 of Experiment 1. After children practiced this procedure with three fractions, they were introduced to the procedure for estimating sums of fractions using fraction strips, as in Part 2 of the individual-fractions-and-sums condition intervention in Experiment 1. They used this procedure to place three unequal-denominator fraction sums on the number line. As in Experiment 1, if a child used the fraction strips incorrectly, the experimenter pointed out the error and guided the child to perform the correct procedure.

Children then played the game with fraction sums. The game was the same as in Part 2 of the intervention in the individual-fractions-and-sums condition of Experiment 1, except that the first phase lasted 4 min, the second phase 5 min, and the third phase 6 min, instead of each phase lasting 4.5 min. This change was made to increase practice time in the later, more difficult phases. As in Experiment 1, if a child used the fraction strips incorrectly, resulting in the monster escaping, the experimenter explained the error before the next trial. Children completed an average of 50.7 trials.

The sums that were shown met the same requirements as in the individual-fractions-and-sums condition in Experiment 1. Also, sums that later appeared in the fraction sum magnitude comparison task on the posttest did not appear during the intervention, nor did sums with the same addends in reverse order.

Whole-number-sums condition. Children were shown whole number strips representing 1, 5, 10, 20, 50, 100, 200, and 500 and taught to use the number strips to find the location of a number on a 0–1,000 number line by concatenating number strips representing the hundreds digit, tens digit, and units digit of the number. Children practiced this procedure with three whole numbers. Next, they were instructed in a procedure for estimating whole number sums by concatenating number strips representing the two addends in each problem. Children used this procedure to place three whole number sums on the number line. If a child used the whole number strips incorrectly, the experimenter guided the child to perform the correct procedure.

Children in this condition then played the game with whole numbers. The game was the same as that employed in the fraction-sums condition, with three exceptions: fraction strips were replaced with whole number strips; the 0–1 number line was replaced with a 0–1,000 number line; and the stimuli were whole number sums. Stimuli were created by generating fraction sums in the same way as in the fraction-sums condition and then converting the fractions to whole numbers roughly equal to the fractions × 1,000 (e.g., 3/5 + 1/7 might be converted to 598 + 145). Rough, rather than exact, equivalents were chosen to minimize children's computation of exact answers (e.g., computation seemed less likely with an addend of 598 than 600). Gameplay consisted of the same three phases, lasting the same lengths of time, as in the fraction-sums condition. As in the fraction-sums condition, feedback following incorrect use of the number strips was given during phase 1. Children completed an average of 46.8 trials.

Assessments. The pretest and posttest each included three tasks, all of which involved addition of fractions with unequal denominators: fraction sum magnitude comparison, fraction sum magnitude comparison with think-aloud protocols, and number line estimation of unequal-denominator sums. The tasks were always presented in that order, so that doing the comparison task with think-aloud could not affect performance on the comparison task without think-aloud and so that doing the number line estimation task could not affect performance on either comparison task. All assessment items are provided in Supplement 2 in the online supplementary materials.

Fraction sum magnitude comparison. Children were shown a fraction addition problem above a 0–1 number line and asked to indicate whether the sum was less than or greater than one. They were instructed not to calculate answers but to imagine where they would mark the answers on the number line, to select "less" if their mark would go before one, and to select "greater" if their mark would go after one. Children could not actually mark the number line. Each child was presented one of two sets of 12 unequal-denominator sums, including three sums less than 0.5 (e.g., 2/8 + 1/9), three sums greater than 0.5 but less than 0.8 (e.g., 4/6 + 1/9), three sums greater than 1.2 but less than 1.5 (e.g., 7/8 + 2/5), and three sums greater than 1.5 (e.g., 9/10 + 6/7). For half of the sums in each set, at least one of the addends was a unit fraction.

Fraction sum magnitude comparison with think-aloud protocols. This task was the same as the fraction sum magnitude comparison task, except that children were asked to think aloud while performing it. The purposes of this task were to assess children's strategies on the fraction sum magnitude comparison task and to test whether their strategies changed after each intervention. Stimuli were two pairs of fraction sums: one pair was 2/10 + 1/8 and 8/9 + 5/6, and the other pair was 2/8 + 1/9 and 9/10 + 6/7. Children completed two trials on the pretest using one pair of sums and two trials on the posttest using the other pair.

Number line estimation of unequal-denominator fraction sums. This task was identical to the corresponding task in Experiment 1.

Analyses. Analyses on all tasks were conducted as with the corresponding tasks in Experiment 1. All reported effects from ANCOVA remained if pretest score was not included as a covariate, if the test version used on the pretest was included as a factor, or both. All effects of condition also appeared as interactions of condition and time of test when the data were analyzed using performance as the dependent variable and time of test as a within-subjects factor.

Results

Table 3 shows performance on all tasks at pretest and posttest in the fraction-sums and whole-number-sums conditions. (These data are presented separated by grade in Supplement 4 of the online supplementary materials).

Fraction sum magnitude comparison. Accuracy improved more in the fraction-sums condition (pretest: 65.7%, posttest: 80.8%, change: 15.1%) than in the whole-number-sums condition (pretest: 77.3%, posttest: 77.8%, change: 0.5%), $F(1, 99) = 8.0$,

Table 3
Mean (Standard Deviation) Pretest and Posttest Performance and Change in Performance by Condition (Experiment 2)

Test measure	Fraction-sums condition	Whole-number-sums condition
Fraction sum magnitude comparison – % correct		
Pretest	65.7 (22.6)	77.3 (21.0)
Posttest	80.8 (21.0)	77.8 (22.7)
Change	15.1 (21.1)	0.5 (16.4)
EMM of change	12.9	3.2
Fraction sum magnitude comparison with think-aloud – % correct		
Pretest	73.5 (30.6)	78.6 (28.9)
Posttest	85.3 (23.0)	79.6 (24.8)
Change	11.8 (36.9)	1.0 (23.9)
EMM of change	10.2	3.0
Number line estimation of unequal-denominator sums – PAE		
Pretest	26.0 (10.9)	21.9 (10.0)
Posttest	10.7 (4.8)	19.0 (9.1)
Change	−15.3 (10.5)	−2.8 (7.0)
EMM of change	−13.8	−4.1

Note. EMM = estimated marginal mean (EMMs were derived from the analyses of covariance described in the main text and are adjusted for the covariate [pretest]); PAE = percent absolute error.

p = .005, η_g^2 = .075. Improvement from pretest to posttest was significant in the fraction-sums condition, $t(51) = 5.2$, $p < .001$, $d = 0.69$, but not in the whole-number-sums condition, $t(50) = 0.21$, $p = .83$, $d = 0.02$.

Accuracy of comparisons to fraction sums greater than one to the number one was of particular interest, because such sums were not presented during the intervention in either condition. In the fraction-sums condition, accuracy improved from pretest to posttest for both sums greater than one (pretest: 64.7%, posttest: 77.2%, change: 12.5%, $t(51) = 2.9$, $p = .005$, $d = 0.43$) and sums less than one (pretest: 66.7%, posttest: 84.3%, change: 17.6%, $t(51) = 4.6$, $p < .001$, $d = 0.58$). Analogous tests were not performed in the whole-number-sums condition because no overall improvement was found in that condition.

Fraction sum magnitude comparison with think-aloud protocols. Change in accuracy from pretest to posttest did not differ between conditions, $F(1, 96) = 2.4$, $p = .12$, $\eta_g^2 = .025$. Across conditions, accuracy increased from 76.0% to 82.5%, $t(99) = 2.1$, $p = .042$, $d = 0.24$.

On the two pretest and two posttest trials where think-aloud protocols were obtained, the protocols were coded as involving estimation strategies if they referenced position on the number line (e.g., "I look at the number line and I picture 9/10 being around here, and 6/7 being around here") or if they referenced the approximate size of the operands (e.g., "2/10 and 1/8 are less than 1/2 so that means that they are going to be less than 1"). The protocols were coded independently by two coders; disagreements occurred on 4% of trials and were resolved through discussion.

The number of children who reported using estimation on at least one of the two test trials on which think-aloud protocols were obtained increased from pretest to posttest in both the fractions-sums condition (pretest: $N = 4$ [8%], posttest: $N = 22$ [43%]), $\chi^2(1) = 16.1$, $p < .001$, and the whole-number-sums condition (pretest: $N = 11$ [22%], posttest: $N = 18$ [37%]), $\chi^2(1) = 5.1$, $p = .023$, as indicated by McNemar's tests. Children's reports of using estimation strategies were quite consistent within the pairs of items on both pretest and posttest: Twelve of the 15 children (80%) who reported estimating on either pretest trial reported doing so on both trials, and 34 of the 40 children (81%) who reported estimating on either posttest trial reported doing so on both trials. Among children who did not report estimating on the pretest (47 in the fraction-sums condition and 38 in the whole-number-sums condition), there was a tendency for more children in the fraction-sums condition than in the whole-number-sums condition to estimate fraction sums at least once on the posttest ($N = 18$ [38%] in the fraction-sums condition vs. $N = 7$ [18%] in the whole-number-sums condition), $\chi^2(1) = 3.1$, $p = .078$.

When children used estimation strategies, they did so effectively: Accuracy was higher among children who reported estimating on at least one of the two trials than among those who did not on both the pretest (96.7% vs. 72.4%), $t(98) = 3.04$, $p = .003$, and the posttest (92.6% vs. 75.8%), $t(98) = 3.61$, $p < .001$. To assess more precisely the relation between changes in strategy use and changes in accuracy, we identified three categories of children: consistent-estimators, who estimated on at least one of the two items on both pretest and posttest ($N = 15$); never-estimators, who estimated on neither item on both the pretest and posttest ($N = 60$); and posttest-estimators, who estimated only on at least one posttest items but neither of the pretest items ($N = 25$). (No participants estimated only on the pretest.) Change in percent correct magnitude comparison judgments from pretest to posttest differed among the three categories, as indicated by an effect of category when it was added as a factor to the ANCOVA on change scores, $F(2, 88) = 4.5$, $p = .013$, $\eta_g^2 = .094$. Table 4 shows percent correct on pretest and posttest and change in percent correct within each category. Accuracy was higher at posttest than at pretest among posttest-estimators, $t(24) = 2.1$, $p = .043$, $d = 0.64$, whereas pretest and posttest accuracy did not differ for consistent-estimators, $t(14) = 0.0$, $p = 1.0$, $d = 0.0$, or never-estimators, $t(59) = 1.0$, $p = .30$, $d = 0.15$.

Number line estimation of unequal-denominator sums. PAE improved by considerably more in the fraction-sums condition (pretest: 26.0, posttest: 10.7, change: 15.3) than in the whole-number-sums condition (pretest: 21.9, posttest: 19.0, change: 2.8), $F(1, 97) = 60.4$, $p < .001$, $\eta_g^2 = .38$. The improvement was

Table 4
Mean (Standard Deviation) Pretest and Posttest Percent Correct and Change in Percent Correct on the Fraction Sum Magnitude Comparison Task With Think-Aloud Among Consistent-Estimators, Never-Estimators, and Posttest-Estimators (Experiment 2)

Test measure	Consistent-estimators	Never-estimators	Posttest-estimators
Pretest	96.7 (12.9)	71.7 (31.0)	74.0 (29.3)
Posttest	96.7 (12.9)	75.8 (25.2)	90.0 (20.4)
Change	0.0 (18.9)	4.2 (30.9)	16.0 (37.4)
EMM of change	17.5	1.4	16.1

Note. EMM = estimated marginal mean (EMMs were derived from the analyses of covariance described in the main text and are adjusted for the covariate [pretest]).

significant both in the fraction-sums condition, $t(50) = 10.4$, $p < .001$, $d = 1.72$ and in the whole-number-sums condition, $t(49) = 2.9$, $p = .006$, $d = 0.29$.

Discussion

In Experiment 2, children in the fraction-sums condition improved substantially not only on the task they encountered during the intervention but also on a transfer task. Apparently, the fraction-sums intervention helped children to gain a flexible understanding of fraction addition that they could apply to tasks other than the one on which they gained experience. Children in an active control condition—the whole-number-sums condition—increased their accuracy of number line estimation of fraction sums, a task that paralleled the estimation of whole number sums on number lines that they practiced, but they showed no improvement on the transfer task. This pattern suggests that the improvements in the fraction-sums condition did not merely reflect test-retest effects or general benefits of playing a game involving number line estimation.

The think-aloud version of the fraction-sum comparison-to-one task proved revealing about the strategies underlying the improved accuracy on these tasks. More children used estimation strategies on the posttest than on the pretest. Moreover, accuracy improved among children who switched from not estimating on the pretest to estimating on the posttest, whereas accuracy did not change among children whose use of estimation did not change from pretest to posttest. These findings suggest that the improved accuracy largely reflected increased use of estimation strategies. Interestingly, use of estimation increased in both conditions, though the increase tended to be greater in the fraction-sums condition. Children in the whole-number-sums condition may have increasingly estimated fraction sums as a result of estimating whole number sums during that intervention or as a result of gaining a better understanding of additive composition of sums.

Like the individual-fractions-and-sums intervention in Experiment 1, the fraction-sums intervention led to a large improvement in number line estimates of unequal-denominator fraction sums. Children showed this improvement despite not having played the game with individual fractions prior to playing it with fraction sums. Moreover, estimates of unequal-denominator fraction sums improved by much more in the fraction-sums condition than in the whole-number-sums condition, although children in both conditions were taught and practiced a procedure for estimating sums on a number line by estimating the first addend and then estimating and adding the second addend. Thus, the greater improvement observed in the fraction-sums than in the whole-number-sums condition did not merely reflect benefits of learning and practicing a procedure for estimating sums on a number line.

General Discussion

Summary of Key Findings

The PFT framework is based on recognition of commonalities in relations between individual numbers and arithmetic sums that are shared by whole numbers and fractions. We hypothesized that this framework provides a basis for interventions that would help children understand both individual fractions and sums of fractions. In the interventions that we created to test this hypothesis, children created visuospatial representations of individual fractions and fraction sums, used these representations to estimate magnitudes by placing marks on a number line, and received feedback on their answers in the form of representations based on the PFT framework. The interventions led to large improvements in performance on tasks assessing understanding of individual fractions and fraction addition. Below, we discuss implications of the findings for teaching and learning about fractions, for children's numerical development, and for mathematics education in general.

Implications for Teaching and Learning About Fractions

Many children experience great difficulty estimating fraction sums, even after prolonged instruction in fraction addition (Braithwaite et al., 2018; Carpenter et al., 1980; Hecht, 1998). The interventions tested in the present study helped children to overcome that difficulty for both equal- and unequal-denominator fraction sums involving denominators from 2 to 10. To our knowledge, these interventions are the first that have been shown to improve children's estimation of fraction sums. Improvement in this ability was not demonstrated in any of the studies identified in a systematic review of fraction interventions for struggling math learners (Roesslein & Codding, 2018).

The ability to estimate fraction sums is important for several reasons. First, estimation provides a pathway for making sense of fraction addition that does not depend on knowing procedures for calculating fraction sums. This pathway could be especially valuable for students who struggle with learning fraction arithmetic procedures, because it may provide them an intuitive sense of what the fraction arithmetic procedures do. Second, skill at estimating fraction sums could facilitate learning fraction addition procedures by enabling children to reject implausible answers and the incorrect procedures that generate such answers (Booth & Siegler, 2008). For example, a common incorrect procedure for adding fractions is to add their numerators and denominators, as in $2/3 + 3/5 = 5/8$, but knowing that $2/3 + 3/5 > 1$ would enable children to reject this procedure and perhaps try a correct procedure for solving the problem. Third, accurate estimation is useful in the many everyday situations where good approximations are sufficient to meet people's goals.

The success of the interventions likely resulted at least in part from the PFT framework, which was the basis for the instruction children received, the procedures they were encouraged to use, and the feedback they received. PFT combines aspects of two prominent interpretations of fractions: the *part-whole interpretation* and the *measurement interpretation* (Kieren, 1976, 1980). According to the part-whole interpretation, a fraction represents a certain number of parts of a whole that is divided into equal-size parts. Consistent with this interpretation, PFT emphasizes that fractions are composed of parts—that is, unit fractions. This aspect of PFT affords a unified approach to representing fractions and fraction sums, because fraction sums are composed of unit fractions just as individual fractions are. PFT also emphasizes the measurement interpretation of fractions by encouraging children to put unit fractions together end-to-end, making length a highly salient feature of the representation. Because length is a relatively transparent analog for numerical magnitude (de Hevia & Spelke, 2010; Lourenco & Longo, 2010), this aspect of PFT encourages attention to magnitude for both individual fractions and fraction sums. The present findings suggest that it can be productive for children to integrate the part-whole and measurement interpretations of fractions.

Implications for Understanding Children's Numerical Development

The present findings extend the integrated theory of numerical development (Siegler & Braithwaite, 2017; Siegler et al., 2011) in several ways. According to this theory, numerical development involves increasingly precise representation of the magnitudes of increasing ranges and types of numbers, including whole numbers and fractions. Most relevant to the present findings, the theory predicts that understanding numerical magnitudes is closely related to understanding arithmetic. This prediction has been supported by studies showing strong correlations between understanding of whole number magnitudes and whole number arithmetic (Booth & Siegler, 2008; Fuchs et al., 2010) and between understanding of fraction magnitudes and fraction arithmetic (Byrnes & Wasik, 1991; Siegler et al., 2011). It has also been supported by experimental studies in which interventions emphasizing accurate representation of numerical magnitudes yielded improved arithmetic learning, again for both whole numbers (Booth & Siegler, 2008; Siegler & Ramani, 2009) and fractions (Dyson, Jordan, Rodrigues, Barbieri, & Rinne, 2018; Fuchs et al., 2013).

In the case of whole numbers, counting provides a mechanism that could underlie associations between numerical magnitude knowledge and arithmetic skill. Children initially count either to assign a number to a quantity, as when counting a set of objects, or to generate the quantity represented by a number, as when counting five fingers to show the number five. Later, preschoolers discover counting-based strategies for adding and subtracting whole numbers (Shrager & Siegler, 1998; Siegler & Jenkins, 1989), such as calculating "3 + 2" by counting "1, 2, 3" and then "4, 5." Because children count to generate individual whole numbers, as well as sums and differences of whole numbers, it seems likely that counting helps children connect magnitudes of individual whole numbers with whole number arithmetic.

The present findings provide evidence for an analogous mechanism that can be used to connect fraction magnitudes with fraction arithmetic—concatenating unit fractions to generate the magnitudes of individual fractions and fraction sums. Thinking of individual fractions as being composed of unit fractions seems to improve children's understanding of fraction addition, as evidenced by the fact that the individual-fractions intervention in Experiment 1 led to improved estimation of both equal- and unequal-denominator fraction sums. For equal-denominator sums, the improvement was as large as that observed in the individual-fractions-and-sums condition. The substantial transfer from individual fractions to equal-denominator sums may reflect the fact that both can be generated by repeatedly adding a unit fraction, just as both whole numbers and whole number sums can be generated by repeatedly adding one whole.

On the other hand, for estimation of unequal-denominator sums, the individual-fractions-and-sums condition led to considerably greater improvement than the individual-fractions condition. Estimating unequal-denominator sums without first converting to a common denominator requires concatenating different unit fractions. Relatively brief practice doing so, in the individual-fractions-and-sums condition, enabled children to solve these more complex problems quite effectively. This complexity has no analogue in the case of whole numbers, for which the unit is always the same (one whole), which probably contributes to the greater ease of understanding whole number than fraction arithmetic. This illustrates a central tenet of the integrated theory: Understanding numerical development requires recognizing both similarities and differences among different types of numbers. One pair of similarities and differences that students might benefit from knowing is that fractions with common denominators, like whole numbers, are composed of the same units, but fractions with different denominators are composed of different units.

Implications for Mathematics Education

The present results inform the interpretation of previous findings regarding how visuospatial representations can be used to improve mathematics instruction. Such representations have been shown to sometimes improve mathematics learning. For example, in Booth and Siegler (2008), first graders who studied whole number addition facts accompanied by number line representations of the addends and sums learned the facts better than children who studied the facts alone. However, children who generated their own number line representations before being shown accurate representations, or who generated their own representations and were not subsequently shown accurate representations, learned no better than children who studied the facts alone. The authors concluded that having children generate their own representations of addends and sums did not improve learning and may have decreased it.

In the present study, asking children to generate number line representations of the magnitudes of fractions and fraction sums was an effective approach. This effectiveness may reflect children being explicitly taught procedures for generating the representations and being provided scaffolding—fraction strips—that led to extremely accurate representations. Children's mean absolute error when estimating unequal-denominator fraction sums with the aid of movable fraction strips was only 1.4% in the individual-fractions-and-sums condition of Experiment 1 and 1.7% in the fraction-sums condition of Experiment 2. These levels of accuracy are superior to the PAEs previously found with university students estimating whole numbers on 0–1,000 number lines (Siegler & Opfer, 2003). The present findings are consistent with Booth and Siegler's (2008) conclusion that "pictorial representations of numerical magnitudes must be accurate to enhance learning"; the findings also show that children can generate accurate representations if provided appropriate scaffolding.

The present findings are also consistent with previous research advocating "concreteness fading" in mathematics and science education. Concreteness fading describes instruction in which concrete representations are presented initially but are subsequently withdrawn to focus learners' attention on underlying structure (Fyfe et al., 2014). The present study implemented this approach by initially presenting children with movable fraction strips that could be aligned along a number line, later making the strips visible but immovable, and finally hiding them. The goal was for children to transition from relying on perception and manipulation of the fraction strips to relying on mental representations. Children's improved performance on posttest assessments, in which fraction strips were not available, indicates that they made the transition successfully. The present study with children joins previous successful implementations of concreteness fading with adults for instruction in combinatorics (Braithwaite & Goldstone, 2013), modular arithmetic (McNeil & Fyfe, 2012), and complex systems concepts (Goldstone & Son, 2005). Concreteness fading provides a promising approach to instruction in other areas as well.

Limitations and Directions for Future Research

Several limitations to the current research are worth noting. The tasks used represent only part of what children need to learn about fractions and fraction arithmetic (CCSSI, 2010). Neither the interventions nor the assessments required children to determine the sizes of unit fractions given the size of the whole; to place fractions or sums of fractions on number lines of varying lengths; to estimate sums involving improper fractions, mixed numbers, or denominators larger than 10; or to calculate exact numeric values of fraction sums. Testing whether PFT facilitates learning in these areas, either alone or in combination with other instructional approaches, seems a useful direction for future research.

Related, the present study demonstrated the effectiveness of PFT only in the context of brief, targeted interventions. Further evaluation of the importance of PFT will require assessing the added value of incorporating PFT into more comprehensive interventions, ideally ones that share PFT's emphasis on magnitudes and its use of the number line as a central conceptual structure. Several such interventions have been developed recently, and each of them has yielded better learning outcomes than control interventions lacking the aforementioned characteristics (Dyson et al., 2018; Fuchs et al., 2013; Saxe, Diakow, & Gearhart, 2013). However, none of these interventions emphasized the idea of representing fraction sums by putting unit fractions together, whereas this idea is central in PFT. Further, none of these interventions have been shown to improve children's estimation of fraction sums. These facts, and the promising results of the present study, suggest that incorporating PFT into existing larger-scale classroom interventions might improve their effectiveness.

Within a larger-scale intervention, PFT might be especially useful for teaching about addition of fractions with unequal denominators, an exceptionally pervasive and persistent source of difficulty for children (Newton, Willard, & Teufel, 2014; Siegler & Pyke, 2013). When beginning this topic, a teacher might ask children to guess a simple sum, say 2/3 + 1/2; many children would likely guess 3/5. The teacher or children might then represent the sum using two 1/3 strips and one 1/2 strip, showing that the sum is larger than one and therefore cannot equal 3/5. Next, the teacher might demonstrate that the sum is unchanged if the two 1/3 strips and one 1/2 strip are replaced by four 1/6 strips and three 1/6 strips respectively, so that the sum equals 7/6. Using this example as a foundation, the teacher could now introduce a symbolic method for calculating the sum by converting the addends to a common denominator.

Another limitation of the present study is that the interventions were administered one-on-one by experimenters working under controlled conditions. It remains to be seen whether the approach embodied in the interventions would be effective when administered to groups of students by teachers in a classroom. Also, the assessments were administered immediately after the interventions, leaving open the question of whether gains resulting from the interventions would be sustained over longer periods.

Another set of limitations involves the samples of students who participated. Participants in both experiments were drawn from schools where more than half of students were Caucasian and fewer than half were eligible for free or reduced-price lunch. It is therefore uncertain whether the present conclusions apply to majority-minority or lower socioeconomic status school populations. Also uncertain is the degree to which the conclusions may be generalized to different instructional contexts, because information about the fractions instruction previously received by participants was not collected during the experiments.

Finally, the present studies included only limited data on individual differences among children that might influence the effectiveness of PFT-based interventions. Information was not collected about domain general cognitive characteristics, such as spatial reasoning and working memory, or about overall math achievement. Because the interventions rely on visuospatial representations, they might be particularly helpful for children who are strong at spatial reasoning, but less effective for children with weak spatial skills. On the other hand, the strong scaffolding of the concreteness fading procedure might make the present instructional procedure more effective than other approaches for children with weaker spatial skills. Similarly, in the third phase of each intervention, children were instructed to maintain and manipulate mental representations of fraction strips that they could not see; working memory limitations might lead some children to struggle with this task, and therefore benefit less from the interventions than students with superior working memories. Finally, the interventions might be more or less effective for children who struggle with mathematics than for those who do not. Future research should collect richer data on individual differences among children to test these possibilities.

Conclusion

Like many others, Booth and Siegler (2008) argued in the context of whole numbers that "arithmetic learning, even in the sense of memorizing answers to unfamiliar problems, is not a rote activity but rather a meaningful one." In the case of fraction arithmetic, this statement may seem more aspirational than descriptive. Children's answers to fraction arithmetic problems routinely violate basic principles of arithmetic, such as the principle that a sum of positive numbers is greater than either addend. Such errors suggest that many children do not understand the meaning of arithmetic operations in the context of fractions.

The present findings, however, suggest that children *can* make sense of fraction arithmetic when instruction helps them acquire a conceptual framework that connects arithmetic to the internal structure of individual fractions. PFT provides such a framework, at least for the most basic arithmetic operation—addition. Future research should explore whether similar approaches can help endow with meaning other arithmetic operations with fractions.

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